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Abstract

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ON THE COMPLETENESS OF THE SYSTEM $\{z^{\lambda_k}\}$ ON CURVES IN THE COMPLEX PLANE

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Let L be an unbounded curve with a finite number of branches going off to infinity (L may consist of several (a finite number of) connected pieces), having no loops and dividing the plane into a finite number of simply connected infinite domains $G_1, G_2, \dots, G_m$. Let L be rectifiable in every finite part of the plane and, if $\sigma(z)$ is the length of the arc of a connected piece, counted from some point of the piece to its point with affix z (far from the initial point we regard $\sigma(z)$ as a single-valued function of $|z|$), then $d\sigma(z) \leq M\, d|z|$, where M is a constant. Suppose that on L a continuous real function $p(z)$ is given such that, for large $|z|$,

$$p(z) \geq p_0(|z|) = p_0(a) + \int_a^{|z|} \frac{\omega(t)}{t}\, dt,$$

where $\omega(t) \geq 0$ and $\omega(t) \uparrow +\infty$. Assume, finally, that each domain G_i ($i = 1, 2, \dots, m$) contains within itself an angle Δ_i of opening π/α_i, $1/2 < \alpha_i < \infty$ (with vertex not necessarily at the origin).

M. M. Dzhrbashian ([1,2]) showed that if under these conditions

$$\int^{\infty} \frac{p_0(r)\, dr}{r^{1+\omega}} = +\infty, \qquad \omega = (\alpha_1, \alpha_2, \dots, \alpha_m),$$

then on L, in the class $L_2[p(z)]$ of functions $f(z)$ defined and measurable on L and such that

$$\int_L e^{-p(z)} |f(z)|^2\, d\sigma < \infty, \tag{1}$$

the system of polynomials is complete in the sense that

$$\inf_{\{Q\}} \int_L e^{-p(z)} |f(z) - Q(z)|^2\, d\sigma = 0, \tag{2}$$

where $Q(z)$ are arbitrary polynomials.

In the present note we are concerned with deriving conditions under which (2) holds for functions $f(z)$ from $L_2[p(z)]$, if in (2) the functions $Q(z)$ are not arbitrary polynomials, but arbitrary finite linear combinations of functions of the system $\{z^{\lambda_k}\}$, where $\lambda_1, \lambda_2, \dots$ are given whole nonnegative numbers. In short, we are concerned with completeness, in the indicated sense, of the system $\{z^{\lambda_k}\}$.

M. M. Dzhrbashian and I. O. Khachatryan ([6]) considered this question in the case when L is a pair of rays, $\arg z = \pm\pi/2\alpha$, and the λ_n are not necessarily integers.

Theorem. Let $\{\mu_n\}$ be the sequence of all positive integers not included in $\{\lambda_k\}$, and let

$$\lim_{n\to\infty} \frac{n}{\mu_n} = \sigma < 1. \tag{3}$$

Suppose further that each angle Δ_i $(i = 1, 2, \dots, m)$ has aperture $\pi/\alpha_i > 2\pi\sigma$, and one of the domains G_i, for example G_1, contains a curvilinear angle P (far from the origin the angles P and Δ_1 coincide) with vertex at the origin (the origin, in general, does not belong to the domain G_1), intersecting every circle $|z| = r$, $b < r < \infty$, in an arc of length $> 2\pi\sigma r$ (in this sense the aperture of the angle P is greater than $2\pi\sigma$). If, for some $\varepsilon_0 > 0$,

$$\int^{\infty} \frac{p_0(r)\,dr}{r^{1+\omega_1+\varepsilon_0}} = \infty, \qquad \omega_1 = \max(\beta_1, \beta_2, \dots, \beta_m), \qquad \frac{\pi}{\beta_j} = \frac{\pi}{\alpha_j} - 2\pi\sigma, \tag{4}$$

then on L the system $\{z^{\lambda_k}\}$ is complete in the sense of (2) ($Q(z)$ being combinations of powers from $\{z^{\lambda_k}\}$ (in the class $L_2[p(z)]$).

The proof of the theorem is based on the simultaneous application of the method of the Cauchy integral transform, used by Dzhrbashian in ([2]), and the method of using differential equations of infinite order.

To prove the theorem, it is necessary to prove that from the conditions

$$\int_L e^{-p(z)}\overline{f(z)} z^{\lambda_k}\,d\sigma = 0, \qquad f(z) \in L_2[p(z)] \quad (k = 1, 2, \dots) \tag{5}$$

it follows that $f(z) = 0$ almost everywhere on L.

Thus, suppose that (5) holds. As in ([2]), consider the function

$$F(w) = \frac{1}{2\pi i} \int_L \frac{e^{-p(z)}\overline{f(z)}\,d\sigma}{z - w}. \tag{6}$$

In each domain G_j it represents some analytic function $F_j(w)$. Since

$$\frac{1}{z-w} = -\frac{1}{w} - \frac{z}{w^2} - \ldots - \frac{z^{n-1}}{w^n} - \frac{z^n}{w^n(z-w)},$$

then, by virtue of (5), in G_j

$$F_j(w) = \sum_{\mu_k<n} \frac{a_k}{w^{\mu_k+1}} - \frac{1}{w^n}\frac{1}{2\pi i}\int_L \frac{e^{-p(z)}\overline{f(z)}\, z^n\, d\sigma}{z-w} = \varphi_n(w) + r_n(w). \quad (7)$$

Let $w = e^t$. In the t-plane the function $F_j(e^t)$ is regular, in particular, in the domain D_j (it is obtained from the angle Δ_j), which for large $\mathrm{Re}(t)$ asymptotically approaches a horizontal strip of width $\pi/\alpha_j > 2\pi\sigma$. It should be noted that $F_1(e^t)$ is regular in the domain Q (it is obtained from the angle P), which contains an entire curvilinear strip of width (in the vertical direction) $> 2\pi\sigma$. Put

$$\prod_{n=1}^{\infty}\left(1 - \frac{t^2}{(\mu_n+1)^2}\right) = \sum_0^{\infty} c_n t^n, \qquad M(y) = \sum_0^{\infty} c_n y^{(n)}(t).$$

By virtue of (3), the operator $M(y)$ has the properties [3]:

1) The operator $M(y)$ is defined at every point t_0 which is the center of a vertical segment of length $> 2\pi\sigma$, on which the function $y(t)$ is regular; moreover, if $y(t)$ is regular in the rectangle $R:\ |\mathrm{Re}(t-t_0)| \leqslant \varepsilon$,

$|\mathrm{Im}(t-t_0)| \leqslant \pi\omega + \varepsilon$, then there exists a constant $N(\varepsilon)$, independent of $y(t)$, such that at the point t_0

$$|M(y)| < N(\varepsilon)\max_{t\in R}|y(t)|. \quad (8)$$

2) $M\left(e^{\pm(\mu_n+1)t}\right) = 0$, whence $M[\varphi_n(e^t)] = 0$.

3) If $y(t)$ is regular in the rectangle R and $M(y) = 0$, then in a neighborhood of the point t_0 the function $y(t)$ is represented by the absolutely convergent series

$$y(t) = \sum c_{\pm n} e^{\pm(\mu_n+1)t};$$

under the additional assumption that $y(t)$ is regular in a domain of type D_j, it follows from this that $y(t)$ is regular in some half-plane $\mathrm{Re}(t) > \alpha$, and under the assumption that $y(t)$ is regular in a domain of type Q, it is regular in the whole plane.

Let E_j be the half-strip $\mathrm{Re}(t) \geqslant \alpha,\ |\mathrm{Im}(t - t_0)| < \pi/2(\omega_1 + \varepsilon_0)$ (the quantities ω_1 and ε_0 occur in (4)), contained in D_j. Since for large $\mathrm{Re}(t)$ the domain D_j approaches a strip of width π/α_j, the distance from the boundary of E_j to the boundary of D_j (in the limit as $\mathrm{Re}(t) \to +\infty$, equal to

$$\frac{\pi}{2\alpha_j} - \frac{\pi}{2(\omega_1 + \varepsilon_0)} > \pi\varepsilon$$

) may be regarded throughout as $> \pi\omega + \varepsilon,\ \varepsilon > 0$. Taking (7) into account, in E_j we obtain

$$\Phi_j(t) \equiv M[F_j(e^t)] = M[r_n(e^t)],$$

whence, by virtue of (8), in E_j

$$|\Phi_j(t)| < N(\varepsilon) \max |r_n(e^{\xi})|, \qquad \xi \in D_j; \qquad \mathrm{Re}(t) - \varepsilon \leqslant \mathrm{Re}(\xi) \leqslant \mathrm{Re}(t) + \varepsilon.$$

Passing again from t to w and putting $\Phi_j(t) = \psi_j(w)$, we obtain that in a certain angle P_j of opening $\dfrac{\pi}{\omega_1 + \varepsilon_0}$

$$|\psi_j(w)| < N(\varepsilon) \max |r_n(\eta)|, \qquad \eta \in \Delta_j; \qquad \frac{|w|}{a} \leqslant |\eta| \leqslant a|w|, \quad a = e^{\varepsilon}.$$

From this point, relying on (4), we can repeat verbatim the arguments on pp. 362-363 of the article ([2]) and become convinced that $\psi_j(w) \equiv 0$. Thus, $M[F_j(e^t)] \equiv 0$. Hence the function $F_j(e^t)$ is regular in some half-plane $\mathrm{Re}(t) > \alpha$ and is represented there by a series in the functions $e^{\pm(\mu_n+1)t}$, while the function $F_j(w)$ is regular for $|w| >$ const and is represented by a Laurent series in powers $w^{\pm(\mu_n+1)}$. Since $F_j(w)$ is bounded (this is clear from (6)) in an angle of opening $> 2\pi\sigma$ (for this angle one may take any angle internal to Δ_j with sides parallel to the sides of Δ_j), it follows, by virtue of (3), that the Laurent series for $F_j(w)$ contains no positive powers and, consequently, the function $F_j(w)$ is regular at ∞. As for the function $F_1(w)$, since $F_1(e^t)$ is regular in Q and, consequently, is an entire function, it is regular everywhere for $|w| > 0$. We shall verify that $F_1(w) = F_2(w) = \dots = F_m(w)$.

Let $\mu(z)$ be the angle formed by the tangent to L at the point z with the positive direction of the real axis. We have

$$F(w) = \frac{1}{2\pi i} \int_L \frac{e^{-p(z)}\overline{f(z)}e^{-i\mu(z)}\,dz}{z - w}.$$

If Γ is a part of L to one side of which the domain G_k adjoins, and to the other side the domain G_s, then, by a known property of the Cauchy-type integral, almost everywhere on Γ

$$e^{-p(z)}\overline{f(z)}e^{-i\mu(z)} = \pm(F_k(z) - F_s(z)),$$

whence

$$\overline{f(z)} = \pm(F_k(z) - F_s(z))e^{p(z)}e^{i\mu(z)}. \tag{9}$$

If it were the case that $F_k(w) - F_s(w) \neq 0$ and, consequently, for large $|w|$

$$F_k(w) - F_s(w) = \frac{\text{const}}{w^p},$$

then a function $f(z)$ of the form (9) would not satisfy condition (1). Hence $F_1(w) = ... = F_m(w)$ and, by virtue of (4), $f(z) = 0$ almost everywhere on L, as was required to prove.

In ([2]), from completeness in the sense of (2), there is derived as a consequence (under the additional assumption that L consists of one connected piece) the completeness of polynomials on L in the sense

$$\inf_{\{Q\}} \max_{z\in L} e^{-p(z)}|f(z) - Q(z)| = 0 \tag{10}$$

in the class $C[p(z)]$ of functions $f(z)$ continuous on L with the property $e^{-p(z)}f(z) \to 0$ as $z \to \infty$, $z \in L$. In exactly the same way, under the conditions of our theorem one can derive the completeness of $\{1, z^{\lambda_k}\}$ on L in the class $C[p(z)]$ in the sense of (10), where $Q(z)$ denotes all possible linear combinations of the functions from $\{1, z^{\lambda_k}\}$.

The completeness of the system $\{1, z^{\lambda_k}\}$ in this sense in the cases when L is either the whole axis $(-\infty, \infty)$, or the half-axis $(0, +\infty)$ (in the latter case the λ_k are not necessarily integers) was considered by S. Mandelbrojt ([5]). M. M. Dzhrbashyan ([4]) considered the same question in the case when $p(z) = |z|^p$ and L is topologically equivalent to the axis $(-\infty, \infty)$ and is situated between two angles of definite openings with vertices at the points 0 and $\alpha > 0$.

In conclusion we give an example indicating the essential nature in the theorem of the requirement that the domain G_j contain an angle of opening precisely $> 2\pi\sigma$, and the domain G_1 a curvilinear angle of opening $> 2\pi\sigma$ with vertex precisely at the origin. Let $\mu_k = kp + p - 1$ $(k = 1, 2, ...)$, where p is an integer ≥ 2. Then $\sigma = \frac{1}{p}$, and λ_k has the form $mp + j$, $j = 0, 1, \dots, p - 2$.

It is not difficult to see that if on the rays $\arg z = \frac{2\pi}{p}s$ $(s = 0, 1, \dots, p - 1)$ one takes points α_s at equal distances from the origin, then the function z^{p-1}

cannot be approximated simultaneously at all the points α_s $(s = 0, 1, \dots, p-1)$, with arbitrary accuracy, by linear combinations of the functions z^{λ_k} with the indicated λ_k. Hence it follows that:

1) The theorem is not true if, as L, one takes a system (connected with one another in some way) of rays $\arg z = \frac{2\pi}{p} s$, $|z| \geq r_s$, with arbitrary r_s. This shows the essential nature of the requirement that the angle Δ_j have opening $> 2\pi\sigma$, since in our case Δ_j has opening $\frac{2\pi}{p} = 2\pi\sigma$.

2) The theorem is not true if L consists of the ray $\arg z = 0$ and an arc of the circle $|z| = r$, $0 \leq \arg z \leq \frac{2\pi}{p}(p-1)$. This shows the essential nature of the requirement that G_1 contain an angle P of opening $> 2\pi\sigma$, since in our case the angle P has opening $2\pi\sigma$.

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Note: Figure translations are in progress. See original paper for figures.