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Abstract

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MATHEMATICS

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GENERAL SOLUTIONS OF DIFFERENTIAL-DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

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Consider the equation

$$\sum_{k=0}^{l} P_k(\partial/\partial x)u(x-h_k) = f(x). \tag{1}$$

Here $x = (x_1, \dots, x_n)$ is a point of the n-dimensional real space $\mathfrak{R}$; $\partial/\partial x = (\partial/\partial x_1, \dots, \partial/\partial x_n)$; P_k are polynomials with constant complex coefficients; $h_k = (h_{k1}, \dots, h_{kn})$ are fixed points in $\mathfrak{R}$, with $h_0 = 0$ and $h_k \neq h_j$ for $k \neq j$. Let A and B be bounded regions in $\mathfrak{R}$, and assume that all $x + h_k \in B$ if $x \in A$. By K_A and K_B we denote the spaces of infinitely differentiable complex-valued functions $\varphi(x)$, equal to zero respectively outside A and outside B, with the usual topology defined by the norms

$$\|\varphi\|_\mu = \sup_{x;\,|k|\le\mu} |D^k\varphi(x)|, \qquad \mu = 0, 1, \dots,$$

where $|k| = \sum_1^n k_\nu$ and

$$D^k = \partial^{|k|}/\partial x_1^{k_1} \cdots \partial x_n^{k_n}.$$

By K'_A and K'_B we denote the spaces of linear continuous functionals over K_A and K_B, respectively. We shall seek the general form of the solution $u(x)$, assuming that $u \in K'_B$, $f \in K'_A$, and $\varphi \in K_A$. In this case equation (1) is understood in the following sense:

$$\left(u(x), \sum \overline{P}_k(-\partial/\partial x)\varphi(x+h_k)\right) = (f(x), \varphi(x)),$$

where the bar denotes replacement of the coefficients in P_k by their conjugate numbers. The main results of the note are readily transferred to systems of m analogous equations with m unknowns.

Let us perform the Fourier transform F (cf. (1), Ch. II and (2), Ch. III). The spaces K_A and K_B then pass into spaces Z_A and Z_B of entire functions defined in the n-dimensional complex space $\mathfrak{C}$,

$$\psi(s) = \psi(\sigma + i\tau) = F[\varphi] = \int \varphi(x) \exp(x, is)\, dx,$$

where $(x, s) = \sum_1^n x_\nu s_\nu$, with a topology in which neighborhoods of zero are the images, under the mapping F, of neighborhoods of zero of the spaces K_A and K_B. The spaces K'_A and K'_B pass into the spaces Z'_A and Z'_B of linear continuous functionals over Z_A and Z_B, acting according to the formula

$$(F[u], F[\varphi]) = (2\pi)^n (u, \varphi).$$

Let $v = F[u]$, $g = F[v]$,

$$Q(s) = \sum P_k(-is) \exp(h_k, is)$$

and

$$\overline{Q}(s) = \sum \overline{P}_k(is) \exp(h_k, -is).$$

Under the Fourier transform, equation (1) becomes the equation

$$Q(s)v = g,$$

i.e.

$$(v, \overline{Q}(s)\psi(s)) = (g, \psi(s)). \tag{2}$$

Every functional u from K'_A or K'_B can be represented in the form (cf. (3), Ch. III, §§ 6–9 and (2), Ch. II, § 4) of a finite sum

$$(u, \varphi) = \sum_{|r| \leq N} \int D^r \varphi(x)\, d\mu_r(x), \tag{3}$$

where $d\mu_r(x)$ are arbitrary measures ((3), Ch. 1, § 1), which it is convenient for us to regard as concentrated in the closure $\overline{C}$ of a fixed bounded domain C containing all $x + h_k$ for $x \in B$. We emphasize that if in equation (1) $l = 0$, i.e. if it is a partial differential equation, then in what follows one may take $A = B = C$. From (3) it follows that every functional v from Z'_A or Z'_B has the form (cf. (2), Ch. III, § 2)

$$(v, \psi) = \int_{\mathfrak{R}} v(\sigma)\psi(\sigma)\, d\sigma, \tag{4}$$

where

$$v(s) = \sum_{|r|\le N} s^r \int_C \exp(x, -is)\, d\mu_r(x). \tag{5}$$

Here, for $\psi \in Z_A$, the entire function $v(s)$ is determined up to the addition of any function $\omega(s)$ of the form (5) for which

$$\int_{\mathfrak{R}} \omega(\sigma)\psi(\sigma)\, d\sigma = 0$$

for all $\psi \in Z_A$. We shall call such an $\omega(s)$ a **zero function**.

For every polynomial $P(s)$ one can construct in $\mathbb{C}$ a contour H, called a **Hörmander staircase** for P, possessing the following properties (see (4) and (2), Ch. II, § 3): 1) $|P(s)| \ge \text{const} > 0$ on H; 2) H lies in some "strip" $|\tau_j| < M$ $(j = 1, \dots, n)$; 3) if a function $V(s)$ is analytic in this strip and decreases sufficiently rapidly in it at infinity, then

$$\int_{\mathfrak{R}} V(\sigma)\, d\sigma = \int_H V(s)\, ds.$$

Let us verify that an analogous contour H can be constructed also for any function $Q(s)$, with a slight modification of 2).

I. Let $l > 0$ and let $P_k(s) \not\equiv \text{const}$ for some k. Performing in (1), if necessary, a nondegenerate real transformation

$$y_k = \sum \alpha_{kj} x_j + \beta_k,$$

one can arrange that: 1) all $P_k(-is)$ have the form

$$c_k s_1^{m_k} + \text{terms of order lower than } m_k \text{ in } s_1,$$

where $c_k = \text{const} \ne 0$; 2) $h_{j1} \ne h_{k1} > 0$. Let h_{l1} be the largest of the numbers h_{k1}. We now define H by the equalities

$$\tau_2 = \dots = \tau_n = 0, \qquad \tau_1 = T(\sigma_1, \dots, \sigma_n).$$

Here T is a function whose values lie between

$$-M \ln(\|\sigma\| + e)$$

and

$$-M \ln(\|\sigma\| + e) - N$$

(where $M > 0$ and $N > 0$ are sufficiently large), such that

$$|c_l| \exp(-h_{l1}\tau_1) - \left| \sum_{k=0}^{l-1} P_k(-is) \exp((h_k, i\sigma) - h_{k1}\tau_1) \right| \ge 1$$

and

$$|P_l(-is)| \ge |c_l|$$

on H.

II. Let $l > 0$, $P_k \equiv \text{const}$ for all k. Then no change of coordinates is needed, and for H one may take the product of n straight lines $\tau_k = \text{const}$, where all constants, except perhaps one, are equal to zero.

III. Let $l = 0$. Then $Q(s)$ is a polynomial.

We pass to the main results. Let H be a fixed Hörmander ladder for the function $Q(s)$, and let $g(s)$ be a fixed function corresponding to the functional $g \in Z'_A$ by formula (4). One can verify that the functional $v \in Z'_B$ then and only then satisfies equation (2) over Z_A, when

$$(v, \psi) = \int_H \frac{g(s) + w(s)}{Q(s)} \psi(s)\, ds \qquad (\psi \in Z'_B), \tag{6}$$

where $w(s)$ is an arbitrary zero function.

Denote by Δ the complement of A in $\overline{C}$. Let the regions A and C be such that Δ is regular ((3), Ch. III, § 9). It can be proved that then every zero function has the form (5), where the integrals must be taken not over $\overline{C}$, but over Δ. Conversely, every function of this form is zero. Thus, one obtains a description of all $v \in Z'_B$ satisfying equation (2) over Z_A, and thereby of all $u \in K'_B$ satisfying equation (1) over K_A. In particular, for $f = 0$

$$(u, \varphi) = \sum_{|r| \le N} \int_\Delta \left\{ \int_H \frac{s^r}{Q(s)} \int \varphi(x) \exp(x - y, is)\, dx\, ds \right\} d\mu_r(y). \tag{7}$$

For each $y \in \Delta$ the functional standing in braces may be regarded as a solution over the space K of all finite infinitely differentiable functions (1^{-3}) of equation (1), in which the right-hand side is a derivative of some order of the δ-function at the point $x = y$, multiplied by a constant.

Let us consider in more detail the case when H lies in the strip $|\tau_j| \le M$ (see II and III above). Let $y \in \mathfrak{A}$ and $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$, where each ε_k is equal to $+1$ or -1. Denote by $L(y, \varepsilon)$ the "coordinate angle with vertex at the point y," i.e. the set of points x of $\mathfrak{A}$ whose coordinates satisfy the inequalities $(x_k - y_k)\varepsilon_k \ge 0$ for all k. Suppose that the set Δ can be divided into a finite number of pairwise nonintersecting parts $\Delta_1, \dots, \Delta_q$ and that to each part Δ_j one can assign a set $\varepsilon^j = (\varepsilon^j_1, \dots, \varepsilon^j_n)$ of $+1$' s and -1' s in such a way that the complement of $\mathfrak{A}_j$ to the set $\bigcup_{y \in \Delta_j} L(y, \varepsilon^j)$ contains the region A.

Let m_k be the highest order of differentiation with respect to x_k in equation (1). Fix arbitrarily: integers $l_j \ge m_j + 2$ ($j = 1, \dots, n$); a Hörmander ladder H for the function $Q(s)$; real numbers b_{jk} sufficiently large in modulus ($> M$, if H lies in the strip $|\tau_j| \le M$, $j = 1, \dots, n$), ($j = 1, \dots, q$; $k = 1, \dots, n$), such that $b_{jk}\varepsilon^j_k > 0$. Consider the function

$$U_{j;\mu}(x) = \int_{\Delta_j} \int_H \frac{\exp(y - x, is)\, ds}{Q(s)(ib_{j1} - s_1)^{l_1} \cdots (ib_{jn} - s_n)^{l_n}}\, d\mu(y). \tag{8}$$

This function is continuously differentiable everywhere at least up to order $(l_1 - 2, \dots, l_n - 2)$ and is an ordinary solution of equation (1) with $f = 0$ for $x \in \mathfrak{A}_j$. From formula (7) it follows that any functional $u \in K'_B$ satisfying equation (1) with $f = 0$ over K_A can be represented in the form of a finite sum of derivatives, in the sense of generalized functions, of ordinary solutions of the form (8) (cf. (3)):

$$(u, \varphi) = \sum_{|r| \le N'} \sum_{j=1}^{q} \int \overline{U_{j;\mu_r}(x)}\, D^r \varphi(x)\, dx. \tag{9}$$

Hence, for equation (1) with $f = 0$, one obtains a new proof of L. Schwartz' s theorem ([3]), Ch. VI, Theorem 29, stating that if all sufficiently smooth solutions are infinitely differentiable (analytic), then all solutions are infinitely differentiable (analytic).

The separability condition on the set Δ given above singles out those domains A for which the representation of the solution in the form (8)–(9) can be obtained without resorting to a change of coordinates or to a decomposition of the domain A itself into parts. This condition is satisfied, in particular, by any convex bounded domain; moreover, for such a domain the set Δ can always be taken to be regular. To split Δ in this case into parts Δ_j, we draw supporting hyperplanes parallel to the coordinate hyperplanes; let $c_1, \dots, c_{2n}$ be the vertices of the circumscribed parallelepiped thus obtained. The parts Δ_j may be formed so that, for each $y \in \Delta_j$, the segment joining y and c_j does not intersect A. In the case where A is a convex polyhedron, it is simplest to split Δ into parts Δ_j by drawing hyperplanes through all its $(n-1)$-dimensional faces. Then each $\mathfrak{A}_j$ will contain the half-space $\mathfrak{A}_j^*$ bounded by one of the hyperplanes drawn; $\bigcap \mathfrak{A}_j^* = A$. From (7) it is clear that every functional $u \in K'_B$ satisfying equation (1) with $f = 0$ on K_A is a finite sum of functionals u_j on K satisfying the equation on functions φ from K that are equal to 0 outside $\mathfrak{A}_j^*$.

If the τ_j are not bounded on H (see I) and if A is the n-dimensional parallelepiped $|x_j| < a_j$ $(j = 1, \dots, n)$, then the result stated at the end of p. 11 holds, with formula (8) replaced by an analogous formula. The case $l = 0$, A an n-dimensional cube, $f = 0$, was considered earlier by the author in ([5]).

In conclusion we note that, without resorting to generalized functions, it is easy to prove the following assertion:

Let A, B, and C be bounded domains, with $\overline{A} \subset B$, and let all $x - h_k \in C$ if $x \in B$. In order that an infinitely differentiable function $u(x)$ in B be, for $x \in \overline{A}$, a solution of equation (1), in which $f(x)$ is an infinitely differentiable function in B, it is necessary and sufficient that

$$u(x) = (2\pi)^{-n} \int_H \frac{[g(s) + h(s)] \exp(x, -is)}{Q(s)} \, ds \qquad (x \in \overline{A}). \tag{10}$$

Here H is the Hörmander ladder for $Q(s)$, and g and h are the Fourier transforms of everywhere infinitely differentiable functions that coincide in $\overline{A}$, respectively, with $f(x)$ and with 0, and are equal to 0 outside C (cf. ([1]), p. 162).

In this note and in ([5]) the author has made use of the valuable advice of G. E. Shilov, the author' s supervisor. The problem of studying the solutions of the partial differential equation $P(\partial/\partial x)u(x) = 0$ with constant coefficients by representing their Fourier transforms in the form of integrals over various contours was posed by I. M. Gelfand; in this general formulation it is, of course, still awaiting its solution. The author expresses his deep gratitude to G. E. Shilov and I. M. Gelfand.

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REFERENCES

[1] I. M. Gelfand, G. E. Shilov, *Generalized Functions and Operations on Them*, Moscow, 1958.
[2] I. M. Gelfand, G. E. Shilov, *Spaces of Basic and Generalized Functions*, Moscow, 1958.
[3] L. Schwartz, *Théorie des distributions*, Paris, 1, 1950; 2, 1951.
[4] F. Treves, C. R., 242, No. 10 (1956).
[5] M. S. Agranovich, *Izv. vyssh. uchebn. zaved.*, Mathematics, No. 4 (5) (1958).

Note: Figure translations are in progress. See original paper for figures.