V. A. KONDRAT'EV

In the present paper we shall consider the equations

1958

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. A. KONDRAT' EV

ON THE OSCILLATION OF SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS OF THE THIRD AND FOURTH ORDERS

(Presented by Academician I. G. Petrovskii, 22 VI 1957)

In the present paper we shall consider the equations

$$y''' + p(x)y = 0; \tag{1}$$

$$y^{\mathrm{IV}} + q(x)y = 0. \tag{2}$$

The functions $p(x)$ and $q(x)$ are continuous for $x \geq x_0$.

A solution of equation (1) or (2) is called **oscillatory** if it has an infinite number of zeros, and **nonoscillatory** if it has only a finite number of zeros. As is known, for an equation of second order two cases are possible: either all solutions oscillate, or all do not oscillate; moreover, the zeros of linearly independent solutions interlace. For equations of higher order, even in the simplest form (1) or (2), these properties are not true. However, under certain restrictions on the coefficients $p(x)$ and $q(x)$, the character of oscillation of the solutions of equations (1) and (2) will be the same as in the case of constant $p(x)$ and $q(x)$, and one can establish theorems on the separation of zeros of the Sturm type for an equation of second order, as well as oscillation and nonoscillation criteria analogous to Kneser' s theorem (^{1}) for an equation of second order.

Theorem 1. *If in equation (1) the function $p(x)$ does not change sign, then between two consecutive zeros of one solution there lie no more than two zeros of any other solution.*

Theorem 2. *If in equation (2) $q(x) \geq 0$, then between two consecutive zeros of some solution there lie no more than four zeros of any other solution.*

For the equation $y^{\mathrm{IV}} + y = 0$ there exists a solution between two zeros of which there lie four zeros of another solution.

The proof of Theorem 1 is based on the fact that if $p(x) > 0$ and $y(x)$ is a solution having a multiple zero at the point x_1, then $y(x) > 0$ for $x < x_1$. In the case $p(x) < 0$ we use the fact that a solution having a multiple zero at the point x increases to the right of x.

At the basis of the proof of Theorem 2 lies the assertion, proved in the work of Švec (^{2}): there does not exist a solution of equation (2), distinct from the identically zero solution, such that

$$y(a) = y'(a) = y(b) = y'(b) = 0,$$

$a \neq b$.

Theorem 3. *If in equation (1)* $p(x)$ *does not change sign, then there exists a nonoscillatory solution.*

In the case $p(x) \geq 0$ there always exists a positive monotonically decreasing solution; if $p(x) \leq 0$, then one can find a positive monotonically increasing solution. If $p(x)$ changes sign, then the assertion of the theorem is not true.

One can give a criterion for all solutions of equation (1) to oscillate. For example, all solutions will be oscillatory if $p(x)$ has-

has the following property: there exists a sequence of intervals (a_n, b_n) such that $b_n - a_n > \alpha > 0$, $\lim_{n\to\infty} a_n = \infty$,

$$\lim_{n\to\infty} \inf_{(a_{2n}, b_{2n})} p(x) = +\infty,$$

$$\lim_{n\to\infty} \sup_{(a_{2n-1}, b_{2n-1})} p(x) = -\infty,$$

and $p(x)$ has only one zero on (b_n, a_{n+1}).

From this, in particular, it follows that all solutions of the equation

$$y''' + x \sin x \, y = 0$$

are oscillatory.

Below we shall give theorems for equations (1) and (2) analogous to Kneser' s theorem for a second-order equation.

Theorem 4. 1) If in equation (1)

$$p(x) \geq \left(\frac{2\sqrt{3}}{9} + \varepsilon_1(x) \right) \frac{1}{x^3},$$

where $\varepsilon_1(x) \geq 0$ and is such that

$$\int^{\infty} \frac{\varepsilon_1(x)}{x}\,dx = \infty,$$

then there exists a fundamental system of solutions such that one of the solutions entering into it tends monotonically to zero, while the other two oscillate and their zeros alternate.

2) If

$$\left(-\frac{2\sqrt{3}}{9} - \varepsilon_2(x)\right)\frac{1}{x^3} \le p(x) \le \left(\frac{2\sqrt{3}}{9} + \varepsilon_3(x)\right)\frac{1}{x^3},$$

where

$$\varepsilon_2(x) \ge 0, \qquad \varepsilon_3(x) \ge 0, \qquad \int^{\infty} \frac{\varepsilon_2(x)\ln x}{x}\,dx < \infty, \qquad \int^{\infty} \frac{\varepsilon_3(x)\ln x}{x}\,dx < \infty,$$

then the solutions of equation (1) are nonoscillatory.

3) If

$$p(x) \le \left(-\frac{2\sqrt{3}}{9} - \varepsilon_4(x)\right)\frac{1}{x^3},$$

where $\varepsilon_4(x) \ge 0$,

$$\int^{\infty} \frac{\varepsilon_4(x)}{x}\,dx = \infty,$$

then there exists a fundamental system of solutions, two of which oscillate and their zeros alternate, while the third tends monotonically to $+\infty$.

Theorem 5. 1) If in equation (2)

$$q(x) \ge \frac{1+\varepsilon_1(x)}{x^4},$$

where $\varepsilon_1(x) \ge 0$,

$$\int^{\infty} \frac{\varepsilon_1(x)}{x}\,dx = \infty,$$

then all solutions of the equation oscillate.

2) If

$$q(x) \leq \frac{-9 - \varepsilon_2(x)}{16x^4},$$

where $\varepsilon_2(x) \geq 0$,

$$\int^{\infty} \frac{\varepsilon_2(x)}{x}\, dx = \infty,$$

then there exists a fundamental system of solutions consisting of two oscillatory and two nonoscillatory solutions; moreover, one of the nonoscillatory solutions tends monotonically to zero, and the other to $+\infty$.

3) If

$$\frac{-9 - \varepsilon_3(x)}{16x^4} \leq q(x) \leq \frac{9 + \varepsilon_4(x)}{16x^4},$$

$$\varepsilon_3(x) \geq 0, \qquad \varepsilon_4(x) \geq 0,$$

$$\int^{\infty} \frac{\varepsilon_3(x)}{x} \ln x\, dx < \infty, \qquad \int^{\infty} \frac{\varepsilon_4(x)}{x} \ln x\, dx < \infty,$$

then the solutions of equation (2) are nonoscillatory.

When condition 1) of Theorem 4 is fulfilled, there are no nonoscillatory solutions except those tending monotonically to zero; hence, if some solution has one zero, then it has an infinite set of zeros.

We note that if in equation (1)

$$p(x) \leq \left(\frac{2\sqrt{3}}{9} + \varepsilon_1(x)\right) \frac{1}{x^3},$$

then there exists a fundamental system of solutions consisting of nonoscillatory solutions. The same can be asserted about equation (2), if

$$q(x) \leq \frac{1 + \varepsilon_1(x)}{x^4}.$$

The nonoscillation of a solution of equation (1) for

$$|p(x)| \leq \frac{1}{x^{3+\alpha}}$$

and of equation (2) for

$$|q(x)| \leq \frac{1}{x^{4+\alpha}}$$

follows from Sobolev' s results (³) on the asymptotic behavior of solutions of linear equations. In the case

$$|p(x)| \leq \frac{\lambda}{x^3}$$

or

$$|q(x)| \leq \frac{\mu}{x^4},$$

one cannot determine from his work whether the solutions will oscillate. Theorems 4 and 5 give the answer to these questions.

Analogous to Theorems 4 and 5 are the following theorems, based on the behavior of $\int^{\infty} px^n dx$ and $\int^{\infty} qx^n dx$.

Theorem 6. If in equation (1) $\int^{\infty} |p| x\, dx = +\infty$, then there exists a fundamental system of solutions consisting of two oscillatory solutions and one nonoscillatory solution.

Theorem 7. If in equation (2) $\int^{\infty} qx^2 dx = \infty$, $q(x) \geqslant 0$, then all solutions are oscillatory; if $\int^{\infty} qx^2 dx = -\infty$, $q(x) \leqslant 0$, then there exists a fundamental system consisting of two oscillatory and two nonoscillatory solutions. In the case $\int^{\infty} |p| x^2 dx < \infty$, $\int^{\infty} |q| x^3 dx < \infty$, the solutions of equations (1) and (2) will be nonoscillatory (this follows from Sobolev' s asymptotic formulas (³)).

In conclusion, we note that the solutions of equation (2) will all be oscillatory if $q(x)$ is a sign-changing function, but its positive part is very large; for example, if there exists a sequence (a_n, b_n), $b_n - a_n > \lambda > 0$, $\lim_{n\to\infty} a_n = +\infty$, $\lim_{n\to\infty} \inf_{(a_n,b_n)} q(x) = +\infty$, or else $b_n - a_n \to +\infty$ as $n \to \infty$, but $\inf_{(a_n,b_n)} q(x) > \alpha > 0$. This condition is satisfied, for example, by the functions $x^\alpha \sin x$ for $\alpha > 0$; $\sin x^\alpha$ for $\alpha < 1$, etc.

Moscow State University
named after M. V. Lomonosov

Received
20 VI 1957

REFERENCES

[1] V. V. Stepanov, *A Course of Differential Equations*, 1953. [2] H. Švec, *Czechoslovak Mathematical Journal*, **4**, issue 1 (1954). [3] I. M. Sobolev, *DAN*, **61**, No. 5 (1948).

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.