I. I. OLKHOVSKII

PHYSICS

1958

SovietRxiv

Abstract

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I. I. OLKHOVSKII

ON A BOUNDARY-VALUE PROBLEM OF GENERALIZED HYDRODYNAMICS

(Presented by Academician N. N. Bogolyubov, 30 VI 1958)

PHYSICS

1. Kinetic boundary conditions were formulated by Maxwell ([1]). The corresponding requirements on the distribution function for stationary motions of a gas were proposed by Epstein ([2]). In paper ([3]), on the same basis, boundary conditions were formulated for the “moments”of the distribution function, valid for nonstationary motions of a gas at a stationary boundary. In the author’s paper ([4]) the apparatus of generalized hydrodynamics was considered, with the help of which, in principle, it is possible to solve boundary-value problems for gases at any value of $\varepsilon = \Delta t_p/\Delta t$ (Δt_p is the relaxation time of a monatomic gas; Δt is the time interval characteristic of the process under consideration). The aim of the present work is to formulate the boundary conditions of generalized hydrodynamics with a nonuniformly moving heat-conducting boundary with accommodation, primarily for the case of one-dimensional linear problems.
2. Let some element of the boundary surface move with velocity $\mathbf{u}_q(t)$. Suppose that a fraction s of the gas atoms incident on this surface element is reflected from it diffusely, and $(1-s)$ specularly, while the diffusely reflected gas atoms obey the Maxwell distribution in velocities (relative to the velocity of the surface element!) with the temperature acquired by the gas particles as a result of accommodation at the surface (the accommodation process is assumed instantaneous). Then the boundary condition is written as follows:

$$f^+(+v_1', v_2', v_3') = (1-s)f^-(-v_1', v_2', v_3') + sA\exp[-(v')^2/2c_s^2]; \qquad v_1' > 0. \quad (1)$$

Here $\mathbf{v}' = \mathbf{v} - \mathbf{u}_q(t)$; $\mathbf{v}$ is the velocity of the gas particles in a stationary coordinate system; v_1' is the projection of the relative velocity on the outward normal to the surface element; v_2', v_3' are the projections on two other axes directed in the plane of the surface element; $f^+ = 0$ if $v_1' < 0$, i.e., f^+ is the part of the distribution function corresponding to reflected gas particles; $f^- = 0$ if $v_1' > 0$, i.e., f^- is the part of the function corresponding to incident particles; consequently, $f = f^+ + f^-$; $c_s^2 = kT_s/m$, T_s is the temperature of the diffusely reflected particles; $T_s - T = \sigma(T_\sigma - T)$; T_σ is the temperature of the surface element; σ is the accommodation coefficient; A is a quantity to be determined, independent of the particle velocity $\mathbf{v}'$.

The boundary condition (1), if the boundary is regarded as stationary, passes into Maxwell's condition. As already noted, condition (1) assumes instantaneous reflection of all gas particles incident on the wall. Consequently, there is no flux of gas particles through the boundary:

$$(u_q)_1 = u_1; \qquad \mathbf{u} = \frac{1}{n}\int \mathbf{v} f\, d\mathbf{v}. \tag{2}$$

From (2) one can determine the quantity A entering condition (1). To do this, we shall find the flux of the number of particles from the wall, multiplying (1) by v_1 and

integrating over the region $v_1' > 0$:

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} v_1 f^+(+v_1', v_2', v_3')\, dv' =$$

$$= (1{-}s)\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} v_1 f^-(-v_1', v_2', v_3')\, dv' {+} sA\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} v_1 \exp\left[-\frac{(v')^2}{2c_s^2}\right] dv'. \tag{3}$$

Let us pass to other variables:

$$\mathbf{c} = \mathbf{v} - \mathbf{u} = \mathbf{v}' - \mathbf{U}, \qquad \mathbf{U} = \mathbf{u}_q - \mathbf{u}. \tag{4}$$

Then (2) is written as follows:

$$U_1 = 0, \qquad c_1 = v_1'. \tag{5}$$

Taking (4), (5) into account, from (3) we obtain:

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (c_1 + u_1') f^+(+c_1, c_2, c_3)\, dc =$$

$$= (1-s)\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (c_1 + u_1') f^-(-c_1, c_2, c_3)\, dc +$$

$$+ sA\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (v_1' + u_1') \exp\left[-\frac{(v')^2}{2c_s^2}\right] dv'. \tag{6}$$

The expansion of f in Hermite polynomials (see (3)) gives, with accuracy up to "moments" of the fourth order:

$$f = f_0 \left\{ 1 + \frac{1}{2} \frac{p_{ij} c_i c_j}{\rho c_e^4} + \frac{1}{6} \frac{S_{ijk} c_i c_j c_k}{\rho c_e^6} - \frac{1}{2} \frac{S_i c_i}{\rho c_e^4} \right\}, \tag{7}$$

where

$$f_0 = \frac{n}{(2\pi)^{3/2} c_e^3} \exp\left[-\frac{c^2}{2c_e^2} \right], \qquad \rho = \int m f \, dv, \qquad c_e = \sqrt{\frac{2}{3} e}, \qquad e = \frac{1}{\rho} \int \frac{mc^2}{2} f \, dc,$$

$$p_{ij} = P_{ij} - p\delta_{ij}, \qquad p = \rho c_e^2, \qquad P_{ij} = \int m c_i c_j f \, dc,$$

$$S_{ijk} = \int m c_i c_j c_k f \, dc, \qquad S_i = \sum_j S_{ijj}.$$

Substituting (7) into (6) and carrying out the corresponding integration, we obtain an expression for A. We give several intermediate integrals:

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f^{\pm}(\pm c_1, c_2, c_3) \, dc =$$

$$= \int_0^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{n}{(2\pi)^{3/2} c_e^3} \exp\left[-\frac{c^2}{2c_e^2} \right] \left\{ 1 + \frac{1}{2\rho c_e^4} [p_{11} c_1^2 + p_{22} c_2^2 + p_{33} c_3^2] \pm \right.$$

$$\left. \pm \frac{1}{6\rho c_e^6} [S_{111} c_1^3 + 3S_{122} c_1 c_2^2 + 3S_{133} c_1 c_3^2] \mp \frac{1}{2\rho c_e^4} S_1 c_1 \right\} dc =$$

$$= \frac{1}{2^{3/2} \pi^{1/2} m c_e^3} \left\{ (2\pi)^{1/2} \rho c_e^3 \mp \frac{1}{3} S_{111} \right\}; \tag{8}$$

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} c_1 f^{\pm}(\pm c_1, c_2, c_3) \, dc = \frac{1}{(2\pi)^{1/2} c_e m} \left(p + \frac{1}{2} p_{11} \right). \tag{9}$$

$$\int_0^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} v_1' \exp\left[-\frac{(v')^2}{2c_s^2} \right] dv' = 2\pi c_s^4, \qquad \int_0^{+\infty} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \exp\left[-\frac{(v')^2}{2c_s^2} \right] dv' = \pi^{3/2} 2^{1/2} c_s^3. \tag{10}$$

Taking (7), (8), (9), (10) into account, we obtain from (6)

$$sA = \frac{s\left[\frac{1}{2}p_{11} + p\left(1 + \frac{\pi^{1/2}}{2^{1/2}}\frac{(u_q)_1}{c_e}\right)\right] - \left(\frac{2-s}{2}\right)\frac{(u_q)_1 S_{111}}{3c_e^2}}{(2\pi)^{1/2} m c_e c_s^4 \left(1 + \frac{\pi^{1/2}}{2^{1/2}}\frac{(u_q)_1}{c_s}\right)}. \tag{11}$$

Thus, the boundary condition (1) with A, determined in (11), is a kinetic boundary condition in the case of a nonuniformly moving heat-conducting boundary with accommodation*.

3. Let us consider the conditions imposed on the hydrodynamic quantities by condition (1). One of them is condition (2). The other requirements** will be the requirements on the heat fluxes $S_{111}, S_{122}, S_{133}$; they are connected with the thermal conductivity at the gas-solid boundary. We obtain these boundary conditions by multiplying (1) by $mc_1c_i^2\,dc$ $(i = 1, 2, 3)$ and integrating over the region $v_1' > 0$.

Carrying out calculations analogous to the preceding ones (see (7), (8), (9), (10)) and taking (4), (5) into account, we obtain:

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} mc_1^3 f^{\pm}(\pm c_1, c_2, c_3)\,dc = \pm\frac{1}{2}S_{111} + \frac{c_e}{(2\pi)^{1/2}}(2p + 3p_{11}); \tag{12}$$

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} mc_1c_2^2 f^{\pm}(\pm c_1, c_2, c_3)\,dc = \pm\frac{1}{2}S_{122} + \frac{c_e}{(2\pi)^{1/2}}\left(p + \frac{1}{2}p_{11} + p_{22}\right); \tag{13}$$

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} mc_1c_3^2 f^{\pm}(\pm c_1, c_2, c_3)\,dc = \pm\frac{1}{2}S_{133} + \frac{c_e}{(2\pi)^{1/2}}\left(p + \frac{1}{2}p_{11} + p_{33}\right); \tag{14}$$

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} c_1^3 \exp\left[-\frac{(v')^2}{2c_s^2}\right]dv' = 4\pi c_s^6; \tag{15}$$

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} c_1c_2^2 \exp\left[-\frac{(v')^2}{2c_s^2}\right]dv' = 2\pi c_s^6\left(1 + \frac{U_2^2}{c_s^2}\right); \tag{16}$$

$$\int_0^{+\infty}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} c_1c_3^2 \exp\left[-\frac{(v')^2}{2c_s^2}\right]dv' = 2\pi c_s^6\left(1 + \frac{U_3^2}{c_s^2}\right). \tag{17}$$

Taking (12), (15) into account, we obtain the boundary condition for S_{111}:

$$S_{111} = \frac{2s}{2-s}\left\{mA4\pi c_s^6 - \frac{c_e}{(2\pi)^{1/2}}(2p + 3p_{11})\right\}; \tag{18}$$

where A is defined in (11). Taking into account (13), (16) and (14), (17) for S_{122} and S_{133}, respectively, we obtain:

$$S_{122} = \frac{2s}{2-s}\left\{mA\,2\pi c_s^6\left(1+\frac{U_2^2}{c_s^2}\right) - \frac{c_e}{(2\pi)^{1/2}}\left(p+\frac{1}{2}p_{11}+p_{22}\right)\right\}; \tag{19}$$

$$S_{133} = \frac{2s}{2-s}\left\{mA\,2\pi c_s^6\left(1+\frac{U_3^2}{c_s^2}\right) - \frac{c_e}{(2\pi)^{1/2}}\left(p+\frac{1}{2}p_{11}+p_{33}\right)\right\}, \tag{20}$$

where A is defined in (11). Since the conditions (19), (20) coincide in form, as unknowns, in accordance with the requirement of one-dimensionality, we may introduce $S_{122} + S_{133}$ or $S_1 = \sum_i S_{1ii}$.* Then from (18), (19), (20) for S_1 we obtain:

$$S_1 = \frac{2s}{2-s}\left\{8\pi c_s^6 mA\left(1+\frac{U^2}{4c_s^2}\right) - \frac{c_e}{(2\pi)^{1/2}}(4p+3p_{11})\right\}. \tag{21}$$

The conditions (2), (18) and (21), in which $U_2 = U_3 = 0$, are the **boundary conditions of generalized hydrodynamics for one-dimensional problems**. The corresponding results of work (3) are obtained from ours if in (2), (11), (21) one sets $(u_q)_t = 0$, $\sigma = 1$, and discards (18).

4. We shall now obtain linear boundary conditions for one-dimensional flows. Let

$$n = n_0 + \Delta n, \quad u = u_0 + \Delta u, \quad p = p_0 + \Delta p,$$

$$p_{11} = (p_{11})_0 + \Delta p_{11}, \quad S_1 = (S_1)_0 + \Delta S_1, \quad S_{111} = (S_{111})_0 + \Delta S_{111}, \tag{22}$$

$$T_s = \{\sigma T_\sigma + T_0(1-\sigma)\} + (1-\sigma)\Delta T,$$

where the subscript 0 refers to the stationary state, and Δ to small nonstationary perturbations. In (22) we set, for definiteness,

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* As we see, when $u_{q1} \sim c_e$, A may tend to infinity. Since in the present work we are primarily seeking to obtain boundary conditions for linear problems, the discussion of this circumstance, which is connected with nonlinear problems, will be carried out later.

** In the present work we consider only those boundary conditions that are significant for one-dimensional problems, with the distribution function having the form (7).

$$u_0 = 0, \quad (p_{11})_0 = 0, \quad (S_1)_0 = 0, \quad (S_{111})_0 = 0, \quad T_\sigma = T_0. \tag{23}$$

Substituting (22), (23) into (11), (18), (21) and neglecting quantities of second order, we obtain

$$A = \frac{n_0}{(2\pi)^{3/2}(c_e)_0^3}\left\{1 + (1-\sigma_1)p' + \frac{1}{2}p'_{11} + \sigma_1 n'\right\}; \tag{24}$$

$$S'_{111} + 2\beta\{\sigma p' + p'_{11} - \sigma n'\} = 0, \quad S'_{11} = \frac{\Delta S_{111}}{\rho_0 (c_e)_0^3}; \tag{25}$$

$$S'_1 + \beta\{4\sigma p' + p'_{11} - 4\sigma n'\} = 0, \quad S'_1 = \frac{\Delta S_1}{\rho_0 (c_e)_0^3}, \tag{26}$$

where

$$n' = \frac{\Delta n}{n_0}, \quad p' = \frac{\Delta p}{p_0}, \quad p'_{11} = \frac{\Delta p_{11}}{p_0}, \quad \beta = \frac{2s}{2-s}\frac{1}{(2\pi)^{1/2}}, \quad \sigma_1 = \frac{1}{2} + 2(1-\sigma).$$

The conditions (2), (25), (26) are the **boundary conditions of generalized acoustics for one-dimensional problems**.

In conclusion I express my deep gratitude to Academician N. N. Bogolyubov for discussing the results.

Moscow State University
named after M. V. Lomonosov

Received
23 VI 1958

CITED LITERATURE

1. J. C. Maxwell, Phil. Trans. Roy. Soc. London, **170**, 231 (1879).
2. P. S. Epstein, Phys. Rev., **23**, 710 (1924).
3. H. Grad, Comm. Pure and Appl. Math., **2**, No. 4, 331 (1949).
4. I. I. Olkhovskii, DAN, **118**, No. 3, 468 (1958).

* The equations of generalized hydrodynamics have an analogous property as well (see (4)).

Note: Figure translations are in progress. See original paper for figures.