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Abstract

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ON A STABLE SYSTEM OF PARTIAL INDICES OF THE HILBERT PROBLEM FOR SEVERAL UNKNOWN FUNCTIONS

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1. Let Γ be a contour consisting of a finite number of simple smooth closed directed curves with continuous curvature, bounding on the left in the complex plane a connected finite domain D_+. Its complement is denoted by D_-. By H we shall denote the set of all functions defined on Γ and satisfying the Hölder condition. By $H_{(n\times 1)}$ (respectively $H_{(n\times n)}$) we shall denote the set of all n-dimensional vector-functions ($n\times n$ matrix functions) with coordinates (elements) from H. The linear set $H_{(n\times n)}$ will be regarded by us as an incomplete linear normed space with the norm defined by

$$\|A\| = n \max |a_{jk}(t)| \qquad \left(A(t) = \|a_{jk}(t)\|_1^n \in H_{(n\times n)}\right),$$

where the maximum is taken over all $t \in \Gamma$ and $j, k = 1, 2, \dots, n$.

Let $A(t) \in H_{(n\times n)}$ be a nonsingular matrix function, and let $\varkappa_1(A) \geq \varkappa_2(A) \geq \dots \geq \varkappa_n(A)$ be the partial indices of the corresponding Hilbert problem ([1],[2])

$$\Phi_+(t) = A(t)\Phi_-(t). \tag{1}$$

We shall call the system of partial indices $\varkappa_j(A)$ ($j = 1, 2, \dots, n$) **stable** if to the matrix $A(t) \in H_{(n\times n)}$ one can assign a $\delta > 0$ such that every matrix $B(t) \in H_{(n\times n)}$ in the δ-neighborhood of the matrix $A(t)$: $\|B - A\| < \delta$, has the same indices as $A(t)$: $\varkappa_j(B) = \varkappa_j(A)$ ($j = 1, 2, \dots, n$). This definition is justified by the following theorem.

Theorem 1*. Let the nonsingular matrix $A(t) \in H_{(n\times n)}$, and let

$$\varkappa = \varkappa(A) = \frac{1}{2\pi}[\arg \det A(t)]_\Gamma.$$

The system of partial indices of the matrix $A(t)$ is stable if and only if

$$\varkappa_1(A) = \dots = \varkappa_r(A) = q + 1; \qquad \varkappa_{r+1}(A) = \dots = \varkappa_n(A) = q, \tag{2}$$

where the integers q, r are determined from the relation $\varkappa = qn + r,\ 0 \le r < n$.

The necessity of the formulated assertion is easily proved by repeating the arguments given in the proof of Theorem 10.2 of the authors' paper (^{3}).

From the same arguments it follows, among other things, that in any neighborhood of an arbitrary nonsingular matrix function $A(t) \in H_{(n \times n)}$ there will always be found matrices $B(t)$ $(\in H_{(n \times n)})$ with a stable system of indices.

The sufficiency of the condition of Theorem 1 follows directly from the following more general propositions.

2. **Theorem 2.** Let the nonsingular matrix function $A(t) \in H_{(n \times n)}$. Then there exists a number $\delta(> 0)$ such that any matrix function $B(t)$ $(\in H_{(n \times n)})$ from the δ-neighborhood of A: $\|B - A\| < \delta$, will be nonsingular and, for every integer p, the inequalities

$$\sum_{\chi_j(A) > p} (\chi_j(A) - p) \ge \sum_{\chi_j(B) > p} (\chi_j(B) - p) \tag{3}$$

will hold.

We shall precede the proof of the theorem with some remarks.

In a natural way, on the contour Γ one defines the Hilbert space $L^{(2)}_{(n \times 1)}$ of n-dimensional vector functions whose coordinates have summable square on Γ. For any $\varphi \in L^{(2)}_{(n \times 1)}$ the singular integral

$$S\varphi = \frac{1}{\pi i} \int_\Gamma \frac{\varphi(\tau)}{\tau - t}\, d\tau, \tag{4}$$

has meaning, defining a certain linear bounded operator S in the space $L^{(2)}_{(n \times 1)}$ (^{4}).

Every solution of the Hilbert problem (1) that vanishes at infinity generates a solution $\varphi(t) = \Phi_+(t) - \Phi_-(t) \in H_{(n \times 1)}$ of the homogeneous singular integral equation

$$(I + A(t))\varphi(t) + (I - A(t))S\varphi(t) = 0, \tag{5}$$

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* After the present note had been submitted, the authors learned of a paper by G. F. Mandzhavidze (^{7}), in which the stability of the system of partial indices is proved for any of its arithmetic structures; this result of (^{7}) is erroneous.

and in this case

$$\Phi(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\varphi(t)}{t-z}\, dt. \tag{6}$$

Conversely, to every solution $\varphi \in H_{(n\times 1)}$ of equation (5), formula (6) corresponds a solution of the Hilbert problem (1) that vanishes at infinity.

It is proved without particular difficulty that if the nonsingular matrix function $A(t) \in H_{(n\times n)}$, then all solutions of equation (5) belong to the set $H_{(n\times 1)}$.

Proof of Theorem 2. Denote by U the operator standing on the left-hand side of equality (5). Note that the dimension $\alpha(u)$ of the subspace $\mathfrak{Z}(u)$ of all zeros of the operator U is always equal to the sum of the positive indices $\chi_j(A)$ of the Hilbert problem (1), which easily follows from the general form of the solutions of this problem ((1), § 127). The sum of the absolute values of all negative indices of problem (1) gives the dimension of the orthogonal complement to the range of the operator U.

Put $A_1(t) = (t-c)^{-p} A(t)$, where c is an interior point of D_+, and p is some integer satisfying the inequalities

$$\chi_n(A) \leq p \leq \chi_1(A). \tag{7}$$

Obviously,

$$\chi_j(A_1) = \chi_j(A) - p \qquad (j = 1, 2, \dots, n).$$

Introduce for consideration the operator U_1 acting in $L^{(2)}_{(n\times 1)}$:

$$U_1\varphi = (I + A_1(t))\varphi(t) + (I - A_1(t))S\varphi(t).$$

Then

$$\alpha(U_1) = \sum_{\chi_j(A) > p} (\chi_j(A) - p), \qquad \beta(U_1) = -\sum_{\chi_j(A) < p} (\chi_j(A) - p).$$

By virtue of the general theorem from (^{5}) (see also Theorems 2.4 in (^{6})), there exists a number $\rho_p > 0$ such that, for any linear operator V_1 acting

in $L^{(2)}_{(n\times 1)}$ and such that

$$\|U_1 - V_1\| < \rho_p, \tag{8}$$

we shall have

$$\alpha(V_1) \leq \alpha(U_1). \tag{9}$$

Denote by δ (> 0) a number smaller than all the quantities

$$\rho_p (1 + \|S\|)^{-1} \min_{t \in \Gamma} |t - c|^p$$

and such that every matrix-function from the δ-neighborhood of $A(t)$ is nonsingular. We shall show that this δ satisfies the requirements of the theorem.

Let, for $B(t) \in H_{(n \times n)}$, the condition $\|A - B\| < \delta$ be fulfilled. Then for any integer p from the interval (7) the matrix-function $B_1(t) = (t-c)^{-p} B(t)$ will be nonsingular, and for it we shall have:

$$\|A_1 - B_1\| < \rho_p (1 + \|S\|)^{-1} \qquad (\varkappa_n(A) \leq p \leq \varkappa_1(A)).$$

Consider the operator V_1, acting in the space $L^{(2)}_{(n \times 1)}$ according to the equality

$$V_1 \varphi = (I + B_1(t))\varphi(t) + (I - B_1(t)) S \varphi(t).$$

Since

$$\|U_1 - V_1\| \leq \|A_1 - B_1\| (1 + \|S\|),$$

the operator V_1 satisfies condition (8). Consequently, for the operator V_1 the relations (9), equivalent to the relations (3), hold. Thus, the inequalities (3) will hold for all integers p from the interval (7). In particular, for $p = \varkappa_1(A)$ and $p = \varkappa_n(A)$ these inequalities give

$$\varkappa_1(A) \geq \varkappa_1(B), \qquad \varkappa_n(A) \leq \varkappa_n(B). \tag{10}$$

Since the δ-neighborhood of the matrix-function $A(t)$ consists of nonsingular matrix-functions, for any matrix-function $B(t)$ from this neighborhood $\varkappa(B) = \varkappa(A)$, and, consequently:

$$\sum_1^n \varkappa_i(B) = \sum_1^n \varkappa_i(A). \tag{11}$$

Hence, from (10) it follows that, for any integer p lying outside the interval $(\varkappa_n(A), \varkappa_1(A))$, the equality sign will occur in relation (3). The theorem is proved.

3. Denote by $\mathfrak{S}_n$ the totality of all possible ordered systems $\{\varkappa_i\}_1^n$ of integers $\varkappa_1 \geq \varkappa_2 \geq \cdots \geq \varkappa_n$. Let $\{\varkappa_j\}_1^n$ and $\{\varkappa_j'\}_1^n$ be two systems from $\mathfrak{S}_n$; we agree to say that the second system is obtained from the first by means of an elementary operation if, for certain integers p and q $(1 \leq p < q \leq n)$,

$$\varkappa_p' = \varkappa_p - 1, \qquad \varkappa_q' = \varkappa_q + 1, \qquad \varkappa_j' = \varkappa_j \quad \text{for } j \neq p, q.$$

We shall further agree to write $\{\varkappa_j\}_1^n > \{\varkappa_j'\}_1^n$ if the system $\{\varkappa_j'\}$ either coincides with the system $\{\varkappa_j\}$, or is obtained from it by successive application of a number of elementary operations. If a system $\{\varkappa_j\}_1^n \in \mathfrak{S}_n$, then we shall call its averaging the system $\{\hat{\varkappa}_j\}_1^n$ defined by the equalities

$$\hat{\varkappa}_1 = \hat{\varkappa}_2 = \cdots = \hat{\varkappa}_r = q + 1, \qquad \hat{\varkappa}_{r+1} = \hat{\varkappa}_{r+2} = \cdots = \hat{\varkappa}_n = q,$$

where the integers q and r are determined from the relations

$$\sum_1^n x_j = qn + r, \qquad 0 \leq r < n.$$

It is easy to see that always

$$\sum_1^n x_j = \sum_1^n \hat{x}_j, \qquad \{x_j\}_1^n > \{\hat{x}_j\}_1^n.$$

It can be shown that, when equality (11) is satisfied, the totality of all relations (3) is equivalent to the fact that

$$\{x_j(A)\}_1^n > \{x_j(B)\}_1^n \quad (> \{\hat{x}_j(A)\}_1^n).$$

It turns out that, however a system of numbers $\{x_j'\}_1^n$ satisfying the condition

$$\{x_j(A)\}_1^n > \{x_j'\}$$

is chosen, in any δ-neighborhood of the nonsingular matrix-function $A(t) \in H_{(n\times n)}$ there exists a nonsingular matrix-function $B(t) \in H_{(n\times n)}$ such that

$$x_j(B) = x_j' \qquad (j = 1, 2, \ldots, n).$$

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Note: Figure translations are in progress. See original paper for figures.

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