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Abstract

Full Text

Mathematics

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On the Stability of Solutions of Integral Equations under Discontinuous Variation of the Kernel

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I. Statement of the problem. Consider the Fredholm equation

$$u(x) - \mu \int_0^1 K(x,s)u(s)\,ds = f(x), \tag{1}$$

for which $\mu = 1$ is an eigenvalue.

Let $\varphi_1(x), \dots, \varphi_n(x)$ be the eigenfunctions corresponding to $\mu = 1$; let $\psi_1(x), \dots, \psi_n(x)$ be the eigenfunctions of the adjoint kernel $K(s,x)$, and, moreover,

$$\int_0^1 f(x)\psi_j(x)\,dx = 0 \qquad (j = 1, 2, \dots, n).$$

Denote by M the set of solutions of equation (1) for $\mu = 1$.

The principal question studied in this note is the following. Put $\mu = 1$ in (1) and replace the interval $(0;1)$ by the interval $(\lambda;1)$, $0 < \lambda < 1$ (or, equivalently, replace $K(x,s)$ by the discontinuous kernel $K_\lambda(x,s)$, coinciding with $K(x,s)$ for $\lambda < s < 1$ and equal to zero for $0 < s < \lambda$). Will the integral equation obtained in this way,

$$u_\lambda(x) - \int_\lambda^1 K(x,s)u_\lambda(s)\,ds = f(x), \tag{2}$$

be uniquely solvable for every sufficiently small λ? If this question can be answered in the affirmative, then it is necessary to determine whether $\lim_{\lambda\to 0} u_\lambda(x)$ exists; if $\lim_{\lambda\to 0} u_\lambda(x) = u_0(x)$ exists, then whether $u_0(x)$ belongs to the family of solutions M, and, finally, what characteristic property singles out $u_0(x)$ from M.* In addition (in Sec. III), we consider discontinuous variation of the kernel in a neighborhood of the straight line $x = s$.

II. Let $\{\varphi_i\}_1^n$ and $\{\psi_i\}_1^n$ be orthonormalized:

$$\int_0^1 \varphi_i(x)\varphi_j(x)\,dx = \int_0^1 \psi_i(x)\psi_j(x)\,dx = \delta_{ij},$$

$$K(x,s) - \sum_{i=1}^n \psi_i(x)\varphi_i(s) = P(x,s),$$

$$(I-P)^{-1} = I + R,$$

* A similar question in the case of operator equations with operators analytically dependent on a parameter w

where

$$Ph = \int_0^1 P(x,s)h(s)\,ds, \qquad Rh = \int_0^1 R(x,s)h(s)\,ds.$$

Then one can show that the integral equation (2) is equivalent to the system of equations

$$\sum_{i=1}^n \gamma_i(\lambda) \int_0^\lambda \varphi_i(s)\Psi_j(s,\lambda)\,ds = -\int_0^\lambda F(s)\Psi_j(s,\lambda)\,ds \tag{3}$$

$$(j = 1, 2, \dots, n),$$

where

$$\gamma_i(\lambda) = \int_\lambda^1 \varphi_i(s)u_\lambda(s)\,ds,$$

$$F(s) = f(s) + \int_0^1 R(s,t)f(t)\,dt,$$

$$\Psi_j(s,\lambda) = \psi_j(s) + \int_0^\lambda R(t,s,\lambda)\psi_j(t)\,dt,$$

and $R(t,s,\lambda)$ is the resolvent kernel corresponding to the equation

$$v(s) + \int_0^\lambda R(t,s)v(t)\,dt = h(s).$$

The further analysis is based on the following auxiliary propositions.

Lemma 1. Let $p_1(x), p_2(x), \ldots, p_n(x)$ be a system of arbitrary functions, continuously differentiable $n-1$ times in a neighborhood of the point $x = 0$. Put

$$P_i(\lambda) = p_i(\lambda) + \int_0^\lambda R(\lambda, t, \lambda) p_i(t)\,dt,$$

$$Q_i(\lambda) = p_i(\lambda) + \int_0^\lambda R(t, \lambda, \lambda) p_i(t)\,dt$$

and denote by $W(x; \omega_1, \ldots, \omega_n)$ the Wronskian of the system of functions $\omega_1(x), \ldots, \omega_n(x)$ at the point x. Then

$$W(0; P_1, \ldots, P_n) = W(0; Q_1, \ldots, Q_n) = W(0; p_1, \ldots, p_n).$$

($K(x, s)$ is assumed sufficiently smooth.)

Lemma 2. Let $p_1(x), \ldots, p_n(x);\ q_1(x), \ldots, q_n(x)$ be two arbitrary systems of functions, continuously differentiable $n-1$ times in a neighborhood of the point $x = 0$. Then

$$\lim_{\lambda \to 0} \frac{1}{\lambda^{n^2}} \det \left\{ \int_0^\lambda p_i(x) q_j(x)\,dx \right\} =$$

$$= -\frac{[1!2! \ldots (n-1)!]^3}{n!(n+1)! \ldots (2n-1)!} W(0; p_1, \ldots, p_n)\, W(0; q_1, \ldots, q_n).$$

The main results of this note may be formulated as the following theorems.

Theorem 1. If the systems of eigenfunctions $\{\varphi_i\}_1^n$, $\{\psi_i\}_1^n$ are locally biorthogonalizable near $x = 0$ (i.e., admit biorthogonalization on $[0; \lambda]$ for all sufficiently small λ), then equation (2) is uniquely solvable for all sufficiently small λ. In particular, if the functions $\{\varphi_i\}_1^n$, $\{\psi_i\}_1^n$ are $n-1$ times continuously differentiable and if, moreover,

$$W(0; \varphi_1, \ldots, \varphi_n) \neq 0, \qquad W(0; \psi_1, \ldots, \psi_n) \neq 0,$$

then the systems of eigenfunctions are locally biorthogonalizable near $x = 0$, and, consequently, for all small λ equation (2) is uniquely solvable.

Theorem 2. Let $f(x)$ be $n-1$ times continuously differentiable, and let the kernel $K(x,s)$ have derivatives

$$\frac{\partial^j K(x,s)}{\partial x^j}, \qquad \frac{\partial^j K(x,s)}{\partial s^j} \quad (j=1,2,\dots,n-1),$$

which are continuous or polar with respect to (x,s). If

$$W(0;\varphi_1,\dots,\varphi_n)\neq 0, \qquad W(0;\psi_1,\dots,\psi_n)\neq 0,$$

then the solution $u_\lambda(x)$ of equation (2) has, as $\lambda\to 0$, a limit $u_0(x)\in M$, characterized by the conditions

$$u_0(0)=u_0'(0)=\dots=u_0^{(n-1)}(0)=0.$$

Theorem 2 contains a number of restrictions connected with the smoothness of the kernel and of the function $f(x)$. If, however, the interval $[0;1]$ is drilled through simultaneously in a neighborhood of a sufficiently large number of points $c_j\in[0;1]$ $(j=1,2,\dots,m;\ m\geq n)$, then these restrictions need not be imposed on the equation.

Let $f(x)$ be continuous, and let the kernel $K(x,s)$ be continuous or polar. Put

$$e_\lambda=\sum_{k=1}^{m}[c_k-\lambda_k',\ c_k+\lambda_k''], \qquad E_\lambda=[0;1]-e_\lambda,$$

where λ_k', λ_k'' are sufficiently small positive numbers, and consider the equation

$$u_\lambda(x)-\int_{E_\lambda}K(x,s)u_\lambda(s)\,ds=f(x). \tag{4}$$

Let us also introduce the matrices

$$\Phi=\left\{\begin{matrix}\varphi_1(c_1) & \cdots & \varphi_1(c_m)\\ \cdot & \cdot & \cdot\\ \varphi_n(c_1) & \cdots & \varphi_n(c_m)\end{matrix}\right\}, \qquad \Psi=\left\{\begin{matrix}\psi_1(c_1) & \cdots & \psi_n(c_1)\\ \cdot & \cdot & \cdot\\ \psi_1(c_m) & \cdots & \psi_n(c_m)\end{matrix}\right\}.$$

Theorem 3. If $\det(\Phi\Psi)\neq 0$, then equation (4) is uniquely solvable for all sufficiently small

$$\lambda=\sum_{k=1}^{m}(\lambda_k'+\lambda_k'').$$

If $\lambda\to 0$ in such a way that all quantities $\lambda_k'+\lambda_k''$ are of one and the same order, then there exists

$$\lim_{\lambda\to 0}u_\lambda(x)=u_0(x)\in M.$$

The solution $u_0(x)$ satisfies the condition

$$U\Psi=0, \tag{5}$$

where U is the row matrix $\{u_0(c_1), u_0(c_2), \dots, u_0(c_m)\}$.

(Condition (5), obviously, uniquely selects $u_0(x)$ from the n-parameter family of solutions M.)

Theorem 3 admits an obvious generalization to the case of multidimensional integral equations.

III. Let $K(x, s)$ be a polar kernel:

$$K(x,s) = \frac{K_0(x,s)}{|x-s|^{1-\alpha}} \qquad (0 < x \le 1),$$

$K_0(x, s)$ is continuous in the square $0 \le x, s \le 1$. We shall assume the function $f(x)$ to be continuous on $[0; 1]$.

Put

$$K(x,s,\lambda) = \begin{cases} K(x,s), & \text{for } 0 \le x,\ s \le 1,\ |x-s| > \lambda; \\ 0, & \text{for } 0 \le x,\ s \le 1,\ |x-s| < \lambda \end{cases}$$

and construct the integral equation*

$$v_\lambda(x) - \int_0^1 K(x,s,\lambda) v_\lambda(s)\, ds = f(x). \tag{6}$$

Theorem 4. If

$$\det \left\{ \int_0^1 K(x,x) \varphi_i(x) \psi_j(x)\, dx \right\} \neq 0, \tag{7}$$

then there exists

$$\lim v_\lambda(x) = v_0(x).$$

Moreover $v_0(x) \in M$ and

$$\int_0^1 K_0(x,x) v_0(x) \psi_j(x)\, dx = 0 \quad (j = 1, 2, \dots, n). \tag{8}$$

Let us note that in the case of the kernel $K(x, s) = K(x, -s)$, (7) is a necessary and sufficient condition for the biorthogonalizability of the systems $\{\varphi_i\}_1^n$, $\{\psi_i\}_1^n$ on the interval $(0; 1)$.

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REFERENCES CITED

1. I. V. Sukharevskii, DAN, 118, No. 3 (1958).

* $K(x, s)$ is the kernel of equation (1), $f(x)$ is its right-hand side.

Note: Figure translations are in progress. See original paper for figures.