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**Abstract**

**Full Text**

**MATHEMATICS**

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# ON THE PROBLEM OF LONGITUDINAL BENDING OF A ROD OF VARIABLE STIFFNESS

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The problem of the longitudinal bending of a rod of variable stiffness $\rho(s)$ leads ([1,2]) to finding solutions of the nonlinear boundary-value problem

$$\frac{d^2y}{ds^2} = -P\rho(s)y(s)\left[1-\left(\frac{dy}{ds}\right)^2\right]^{1/2}, \qquad y(0)=y(1)=0, \tag{1}$$

where $P$ is the load; $y(s)$ is the deflection (see the drawing in article ([3])). After the substitution

$$\frac{d^2y(s)}{ds^2} = -\varphi(s) \tag{2}$$

the problem reduces to finding nonzero solutions of the nonlinear integral equation

$$\varphi(s) = P\rho(s)\int_0^1 K(s,t)\varphi(t)\,dt\cdot\left\{1-\left[\int_0^1 \frac{\partial K(s,t)}{\partial s}\varphi(t)\,dt\right]^2\right\}^{1/2}, \tag{3}$$

where

$$K(s,t) = \begin{cases} s(1-t), & \text{for } s\le t,\\ t(1-s), & \text{for } t\le s.\end{cases} \tag{4}$$

Equation (3) was investigated by the methods of nonlinear functional analysis by M. A. Krasnosel' skii ([2]) and, more completely, by I. A. Bakhtin and M. A. Krasnosel' skii ([3]). In particular, they showed that, for loads $P$ from a certain interval $(P_1, P_1+h^2)$ ($P_1$ is Euler' s critical load, which, by virtue of M. A. Krasnosel'skii's theorems on bifurcation points, can be found from the linearized equations), problem (1) has a unique positive solution $y(s) = y(s;P)$, which, together with its second derivative with respect to $s$, depends continuously on $P$, and

$$\lim_{P\to P_1} \max_s |y(s;P)| = \lim_{P\to P_1} \max_s |y''(s;P)| = 0 \tag{5}$$

and, for $P^* < P^{**}$,

$$y(s;P^*) < y(s;P^{**}); \qquad y''(s;P^*) > y''(s;P^{**}) \quad (0 < s < 1). \tag{6}$$

In the present article, using the methods developed by P. S. Uryson ([4]), we show that positive solutions of equation (3) can be obtained by the method of successive approximations with a suitable zero approximation. In addition, Uryson’ s method makes it possible to indicate certain additional properties of the solutions of equation (3).

1. Denote by $\alpha$ and $\beta$ the least eigenvalues (they are positive by virtue of Jentzsch’ s theorem ([5,2])) of the kernels

$$P(s,t) = \rho(s)K(s,t), \qquad Q(s,t) = \rho(s)K(s,t)\sqrt{4s - s^2}$$

and by $P(y)$ the least eigenvalue of the kernel

$$F[s,t;y] = \rho(s)K(s,t)\frac{\sqrt{4-(1-2x)^2y^2}}{2}.$$

From arguments essentially repeating the arguments of Uryson ([4]), it follows that for every $P_0 \in (\alpha, \beta)$ there is a $y_0$ such that $P(y_0) = P_0$. Let $\varphi_0(s)$ be the eigenfunction of the kernel $F[s,t;y_0]$, satisfying the inequality $\varphi_0(s) < y_0$. Define a sequence of continuous functions $\varphi_n(s)$ by the equalities

$$\varphi_n(s) = P_0\rho(s)\int_0^1 K(s,t)\varphi_{n-1}(t)\,dt \cdot \left\{1 - \left[\int_0^1 \frac{\partial K(s,t)}{\partial s}\varphi_{n-1}(t)\,dt\right]^2\right\}^{1/2}. \tag{7}$$

**Theorem 1.** *The sequence (7) increases monotonically and converges uniformly to a positive solution on* $(0,1)$ *of equation (3).*

To prove this theorem, one constructs the auxiliary sequence of functions

$$\psi_0(s) = \varphi_0(s),$$

$$\psi_n(s) = P_0\rho(s)\int_0^1 K(s,t)\left\{1-\left[\int_0^1 K_s'(s,t)\varphi_0(t)\,dt\right]^2\right\}^{1/2}\psi_{n-1}(t)\,dt+\varphi_0(s). \quad (8)$$

The uniform convergence of the sequence (8) follows from the general theory of linear Fredholm integral equations. The uniform convergence of the sequence (7) follows from the inequalities proved,

$$\varphi_n(s)-\varphi_{n-1}(s)<\psi_n(s)-\psi_{n-1}(s).$$

The monotonicity of the sequence (7) can be shown, for example, with the aid of Lemma 3 from the article ([3]).

2. By studying the sequence (7) and applying P. S. Uryson' s method, one can obtain new proofs of the above assertions of I. A. Bakhtin and M. A. Krasnoselskii on the properties of solutions of problem (1). Among the new results let us note the following proposition.

**Theorem 2.** *The solution $y(s) = y(s;P)$ of problem (1), together with its second derivative $y''_{s^2}(s;P)$, is a continuously differentiable function of the load $P$.*

The proof of this theorem, as well as the complete proof of theorem 1 ([6]), contains cumbersome calculations and therefore cannot be presented in the present article.

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*Note: Figure translations are in progress. See original paper for figures.*