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Abstract

Full Text

Mathematics

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SOME PROPERTIES OF SINGULAR INTEGRALS OF CAUCHY TYPE AND THEIR APPLICATIONS

(Presented by Academician I. N. Vekua on 14 IV 1958)

Let us take a function $f(t)$, defined on a simple closed Lyapunov curve L, situated in the complex plane with the origin inside L, and consider the integral $Sf = \frac{1}{\pi i}\int_L \frac{f(t)\,dt}{t-t_0}$, $t_0 \in L$, understood as an improper integral in the sense of the Cauchy principal value. Introduce the classes of continuous functions $H(p+\alpha, L)$, where p is an integer, $p \geq 0$, $0 \leq \alpha \leq 1$. A function $f(t) \in H(p+\alpha, L)$ if its p-th derivative $f^{(p)}(t)$ satisfies on L the condition

$$f^{(p)}(t_1) - f^{(p)}(t_2) = O\left(|t_1 - t_2|^{\alpha}\right), \qquad t_1,\ t_2 \in L.$$

The well-known theorem of Plemelj-Privalov ([1]) states that from $f \in H(p+\alpha, L)$ it follows that $Sf \in H(p+\alpha, L)$, and conversely, if $0 < \alpha < 1$. For $\alpha = 0, 1$ this is no longer so.

The present note is devoted to the study of the limiting cases of the Plemelj-Privalov theorem ($\alpha = 0, 1$). The results obtained are then applied to the question of the structural properties of the solution of the Riemann boundary-value problem and of the singular integral equation of normal type.

Let $Z(p, L)$ be the class of functions defined on L whose p-th derivative satisfies the relation

$$f^{(p)}[t,(s+h)] + f^{(p)}[t(s-h)] - 2f^{(p)}[t(s)] = O(h),$$

where s is the arc abscissa of the point $t(s) \in L$. Denote by $\rho_n(f, L)$ the magnitude of the best approximation of the function f, given on L, by means of polynomials of degree n in the positive and negative powers of t:

$$\rho_n(f, L) = \sup_{P_n \in H_n} \inf_{t \in L} |f(t) - P_n(t)|,$$

where H_n is the totality of all polynomials of the form $\sum_{-n}^{n} a_k t^k$.

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Theorem 1. If $f \in Z(p, L)$, then $Sf \in Z(p, L)$, and conversely.

Theorem 2. If $f \in Z(p, L)$, then $\rho_n(f, L) = O(n^{-p-1})$, and conversely.

Theorem 3. If $\rho_n(f, L) = O(n^{-p-\alpha})$, then $\rho_n(Sf, L) = O(n^{-p-\alpha})$, $0 < \alpha \leq 1$, and conversely.

Theorem 4. If $f \in C$, i.e., is continuous on L, then $\exp(Sf) \in L_p$, $p > 0$, i.e., is summable on L with any power $p > 0$.

The proof of all these theorems is based on the representation

$$Sf = \frac{1}{\pi i} \int_{|w|=1} \frac{f[\psi(w)]}{w - w_0} \, dw + \frac{1}{\pi i} \int_{|w|=1} f[\psi(w)] \left[\frac{\psi'(w)}{\psi(w) - \psi(w_0)} - \frac{1}{w - w_0} \right] dw,$$

in which $t = \psi(w)$ are the boundary values of a function conformally mapping the exterior of the circle $|w| = 1$ onto the exterior of L, $t_0 = \psi(w_0)$, and also to the corresponding (when L is the circle: $|w| = 1$) or similar results of the authors ([2–4]).

Let us now consider the Riemann boundary-value problem ([1,5,6]): to find a piecewise-holomorphic function of the form $\frac{1}{2\pi i} \int_L \frac{\varphi(\tau)\, d\tau}{\tau - z}$, whose angular boundary values φ^+ and φ^-, respectively from inside and from outside L, satisfy the condition

$$\varphi^+ = G\varphi^- + f \tag{1}$$

almost everywhere on L, where G is continuous, and $f \in L_r$, $r > 1$, on L, the index $\varkappa = \frac{1}{2\pi}[\arg G]_L = 0$.

We shall regard two functions as identical if their values coincide almost everywhere on L. Following the reasoning of F. D. Gakhov ([5]), represent G in the form $G = \psi^+/\psi^-$, where

$$\psi^{\pm} = \exp\left[\pm\frac{1}{2}\ln G + \frac{1}{2}S(\ln G)\right] \tag{2}$$

and $(\psi^{\pm})^{-1}$, by virtue of Theorem 4, belongs to L_p for every $p > 0$. After this, relation (1) is given the form

$$\frac{\varphi^+}{\psi^+} - \frac{\varphi^-}{\psi^-} = \frac{f}{\psi^+}. \tag{3}$$

By Hölder' s inequality $\frac{f}{\psi^+} \in L_{r-\varepsilon}$, where ε is any arbitrarily small positive number. Solving problem (3) (see ([6])), we obtain

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$$\varphi^{\pm} = \psi^{\pm} \left[\pm \frac{1}{2} \frac{f}{\psi^{+}} + \frac{1}{2} S \left(\frac{f}{\psi^{+}} \right) \right]. \tag{4}$$

Thus, under the assumptions made, there exists a unique solution of the Riemann problem belonging to $L_{r-\varepsilon}$.*

By modifying and generalizing the above arguments, one can similarly investigate the Riemann problem when the index $\varkappa \neq 0$ and when $f(t)$ is a piecewise-continuous function with a finite number of discontinuities of the first kind. The results obtained can also be applied to the study of a general singular integral equation of normal type.

From Theorem 3 and formulas (2), (4) it follows: if $\rho_n(G, L)$ and $\rho_n(f, L)$ have order $n^{-p-\alpha}$ (this will be the case if, for example, G and f belong to $H(p+\alpha, L)$), then $\rho_n(\varphi^{\pm}, L)$ also have order $n^{-p-\alpha}$. This last assertion may find application in investigating the rate of convergence of an approximate solution of the singular integral equation ([7]).

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CITED LITERATURE

[1] N. I. Muskhelishvili, *Singular Integral Equations*, Moscow–Leningrad, 1946.
[2] A. Zygmund, Duke Math. J., **12**, 47 (1945).
[3] V. I. Smirnov, Math. Ann., **107**, 313 (1932).
[4] S. Ya. Alper, Izv. Akad. Nauk SSSR, Ser. Math., **9**, No. 6 (1955).
[5] F. D. Gakhov, Matem. sbornik, **2**, No. 4 (1937).
[6] B. V. Khvedelidze, Tr. Tbilisi Math. Inst., **23** (1956).
[7] V. V. Ivanov, DAN, **114**, No. 5 (1957).

* I. B. Simonenko showed that in the case $r = 2$ in the conclusion formulated, ε can be omitted.

Note: Figure translations are in progress. See original paper for figures.

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