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Abstract

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ON THE TORSION OF CYLINDRICAL SHAFTS WITH CIRCULAR GROOVES

(Presented by Academician A. A. Dorodnitsyn, 6 XII 1957)

The torsion of shafts of variable cross-section is described by the system of equations

$$\frac{\partial \varphi}{\partial r} = \frac{1}{r^3}\frac{\partial \psi}{\partial z}, \qquad \frac{\partial \varphi}{\partial z} = -\frac{1}{r^3}\frac{\partial \psi}{\partial r}, \tag{1}$$

where r and z are cylindrical coordinates; φ and ψ are functions subject to the corresponding boundary conditions.

In the case of an infinite cylindrical shaft with an annular groove of circular shape, under the condition that the lateral surface of the shaft is free of external forces, the function ψ is subject to the boundary conditions

$$\psi|_{r=0} = 0, \qquad \psi|_{L} = \frac{M}{2\pi}, \tag{2}$$

where M is the total twisting moment; $r = 0$ and L are streamlines bounding the axial section of the shaft G (Fig. 1). The main interest here is the determination of the magnitude of the maximum stresses. An exact solution of this problem has not been obtained, and the known approximate solutions are given without indicating the magnitude of the error ([1,2]).

In the present work, by means of the method of majorant domains ([3]), a solution of the stated problem is given in the form of fairly simple formulas which determine the maximum stresses with a sufficient degree of accuracy and indicate the limits of variation of the error.

Let R be the radius of the shaft, ρ the radius of the groove, and α its depth ($\alpha \leqslant 2\rho$, see Fig. 1).

The upper estimate of the magnitude of the stress vector $\mathbf{V}$ is determined by the inequality ([3])

$$\max|\mathbf{V}| \leqslant \frac{2M}{\pi R_0^3}\Pi(\mu), \tag{3}$$

Fig. 1

Figure 1: Fig. 1

where

$$\Pi(\mu) = \frac{3}{4\sqrt{2}} \frac{\sqrt{\mu^2+1}}{\mu} \frac{3\mu^2+1}{2\mu^2+1}, \tag{4}$$

$$\mu = \frac{1}{\sqrt{1+R_0/\rho}}. \tag{5}$$

Fig. 1

To determine the lower estimate of the sought quantity, we construct a majorant domain $\widetilde{G}$, replacing in the domain G the boundary streamline L (Fig. 1)

some other boundary line L'. We choose the latter in such a way that it lies inside the domain G, passes through the point C (see Fig. 1), has curvature $1/\rho$ at this point, and, as $z \to \pm\infty$, approaches asymptotically the straight line $r = R$. An analytic expression for such a curve L' ($\psi = \text{const}$) can be constructed by means of a linear combination of two particular solutions of the system of equations (1):

$$\psi = k_1\psi_1 + k_2\psi_2, \tag{6}$$

where

$$\psi_1 = \frac{\sin^3\theta}{3} - \sin\theta + \frac{2}{3}, \qquad \psi_2 = r^4, \tag{7}$$

$$\theta = \arg\xi, \qquad \frac{l}{2}\left(\xi + \frac{1}{\xi}\right) = r + iz, \qquad l = \text{const} > 0, \tag{8}$$

$$k_1 = \frac{3M}{2\pi} \frac{1-\delta^4}{(\mu+2)(1-\mu^2)}, \qquad k_2 = \frac{M}{2\pi R^4}, \tag{9}$$

$$\delta = \frac{R_0}{R}. \tag{10}$$

In accordance with the general formula for the magnitude of the stress vector,

$$\left|\tilde{\mathbf{V}}\right|_{L'} = \left|\frac{1}{r^2}\frac{d\psi}{dn}\right| \tag{11}$$

Fig. 2

Figure 2: Fig. 2

where $\mathbf{n}$ is the right-hand normal to the streamline $\psi = \text{const}$, we obtain

$$\left|\tilde{\mathbf{V}}\right|_C = \frac{2M}{\pi R_0^3}\left\{\Pi_1(\mu)(1-\delta^4)+\delta^4\right\}, \tag{12}$$

where

$$\Pi_1(\mu) = \frac{3}{4}\frac{(1+\mu)^2}{\mu(\mu+2)}. \tag{13}$$

Fig. 2

Comparing (3) and (12), we obtain rigorously justified estimates of the sought magnitude of the maximum stresses from above and below,

$$\frac{2M}{\pi R_0^3}\left\{\Pi_1(\mu)(1-\delta^4)+\delta^4\right\} \leq \max|\mathbf{V}| \leq \frac{2M}{\pi R_0^3}\Pi(\mu). \tag{14}$$

The arithmetic mean of the upper and lower estimates (14) gives the following simple formula for the approximate determination of the magnitude of the maximum stresses:

$$\max|\mathbf{V}| \approx \frac{M}{\pi R_0^3}\left\{\Pi(\mu)+\Pi_1(\mu)(1-\delta^4)+\delta^4\right\}. \tag{15}$$

The limits of variation of the error of this approximate formula are determined by inequalities (14). The relative error ∇ can be written in the form

$$\nabla = \frac{1-F}{1+F}, \tag{16}$$

where

$$F = \frac{\sqrt{2}}{\sqrt{\mu^2+1}}\,\frac{2\mu^2+1}{(\mu+2)(3\mu^2+1)}\left\{(1+\mu)^2+\frac{\delta^4}{3}(\mu-1)(\mu+3)\right\}. \tag{17}$$

Using equalities (16) and (17), it is easy to calculate the relative error as a function of the shaft parameters R_0/R and ρ/R_0. The results of such calculations are given in Table 1.

Table 1

R_0/R, from	R_0/R, to	ρ/R_0, from	ρ/R_0, to	$\Delta = \frac{1-F}{1+F}$, %
0	0,50	1,3	∞	0,5
0	0,25	0,33	1,3	1,26
0,5	0,75	1,3	∞	2,5
0,25	0,50	0,33	1,3	3,8
0	0,25	0,066	0,33	5,2
0,5	0,75	0,33	1,3	5,5
0,75	1,00	1,3	∞	5,6
0,25	0,50	0,066	0,33	6,7
0,5	0,75	0,066	0,33	14
0,75	1	0,33	1,3	16,3
0	0,25	0	0,066	17,3
0,25	0,5	0	0,066	20,2
0,5	0,75	0	0,066	34
0,75	1	0,066	1,3	64
0,75	1	0	0,066	100

Carrying out a general analysis of the data presented in Table 1, we conclude that formula (15) determines the maximum stresses with a sufficient degree of accuracy, especially for the most common cases of shafts of variable cross section.

In conclusion, we note that the approximate formula (15) and the strict inequalities (14) remain valid not only for annular grooves of circular form, but also for certain other cases, for example, for the case of a shaft with an axial section shown in Fig. 2. Moreover, for the shaft indicated in Fig. 2, the results of Table 1 hold.

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Note: Figure translations are in progress. See original paper for figures.