MATHEMATICS

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Abstract

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MATHEMATICS

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ON THE BOUNDEDNESS OF SOLUTIONS OF BOUNDARY VALUE PROBLEMS FOR LINEAR ELLIPTIC SYSTEMS OF FIRST ORDER IN THE PLANE

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Consider the boundary value problem

$$\alpha u + \beta v \,\big|_L = 0$$

for the system of equations of elliptic type

$$\begin{aligned} a_{11}u_x + a_{12}u_y + b_{11}v_x + b_{12}v_y + c_{11}u + c_{12}v = g_1, \\ a_{21}u_x + a_{22}u_y + b_{21}v_x + b_{22}v_y + c_{21}u + c_{22}v = g_2 \end{aligned}$$

in some simply connected domain G with boundary L; $\alpha(t)$, $\beta(t)$ are functions given on L, Hölder-continuous; $\alpha^2+\beta^2 = 1$. This problem can always be reduced to the following form ([1]):

$$\frac{\partial w}{\partial \bar{z}} + \mu_1(z)\frac{\partial w}{\partial z} + \mu_2(z)\frac{\partial \bar{w}}{\partial \bar{z}} + a_1(z)w + a_2(z)\bar{w} = g(z); \tag{1}$$

$$\mathrm{Re}\{z^{-n}w(z)\}\,\big|_\Gamma = 0; \tag{2}$$

$$z = x + iy; \quad w = u + iv; \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right); \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right);$$

Γ is the boundary of the disk D ($|z| \leqslant 1$);

$$n = \frac{1}{2\pi}[\arg(\alpha + i\beta)]_L$$

is the index of our problem (*). We shall assume that μ_1, μ_2 are measurable functions in the disk D.

The ellipticity condition for the system written in the form (1) is given by the estimate

$$|\mu_1(z)| + |\mu_2(z)| \leqslant \mu_0 < 1. \tag{3}$$

With respect to the remaining coefficients we shall assume:

$$\begin{gathered} a_1(z),\ a_2(z),\ g(z) \in L_p(D), \quad p > 2; \\ \|a_1\|_{L_p},\ \|a_2\|_{L_p},\ \|g\|_{L_p} \leqslant K. \end{gathered} \tag{4}$$

We shall investigate our problem in the following settings:

Case of nonnegative index $n \geqslant 0$.

Problem 1. Find a function $w(z) \in W_p^{(1)}(D)$, $p > 2$, which satisfies equation (1) and the boundary conditions

$$\operatorname{Re}\{z^{-n} w(z)\}\,\big|_{\Gamma} = 0, \qquad \int_{\Gamma} z^{-k} w(z)\, ds = 0 \quad (k = 0, 1, \dots, 2n). \tag{5}$$

The case of negative index $n < 0$.

Problem 2. Find a function $w(z) \in W_p^{(1)}(D)$, $p > 2$, satisfying equation (1), and $2|n| - 1$ real constants $\lambda_0, \lambda_{\pm 1}, \dots, \lambda_{\pm|n|-1}$ such that on the boundary the condition holds

$$\operatorname{Re}\{z^{-n} w(z)\}|_{\Gamma} = \operatorname{Re}\left\{\lambda_0 + \sum_{k=1}^{|n|-1} (\lambda_k + i\lambda_{-k}) z^k\right\}. \tag{6}$$

By the method that we used in [1], Problems 1 and 2 can be reduced to the equivalent singular integral equation

$$\rho + \mu_1 S_n \rho + \mu_2 \overline{S}_n \rho + a_1 T_n \rho + a_2 \overline{T}_n \rho = g, \tag{7}$$

which is equivalent to an equation of Fredholm type, and whose corresponding homogeneous equation has no solutions different from zero [1].

We denoted:

for $n \geq 0$,

$$T_n \rho = -\frac{1}{\pi} \iint_D \left[\frac{\rho(\zeta)}{\zeta - z} + \frac{z^{2n+1}\overline{\rho(\zeta)}}{1 - z\overline{\zeta}}\right] dT_\zeta, \tag{8}$$

$$S_n\rho = \frac{\partial}{\partial z}T_n\rho = -\frac{1}{\pi}\iint_D \left[\frac{\rho(\zeta)}{(\zeta-z)^2} + \frac{z^{2n+1}\overline{\zeta}\,\rho(\overline{\zeta})}{(\zeta-z\overline{\zeta})^2}\right] dT_\zeta -$$

$$-\frac{2n+1}{\pi}\iint_D \frac{z^{2n}\rho(\overline{\zeta})}{1-z\overline{\zeta}}\,dT_\zeta = S_n^0\rho + T_n^0\rho; \tag{9}$$

for $n < 0$,

$$T_n\rho = -\frac{1}{\pi}\iint_D \left[\frac{\rho(\zeta)}{\zeta - z} + \frac{\overline{\zeta}^{\,2|n|-1}\overline{\rho(\zeta)}}{1-z\overline{\zeta}}\right] dT_\zeta, \tag{10}$$

$$S_n\rho = \frac{\partial}{\partial z}T_n\rho = -\frac{1}{\pi}\iint_D \left[\frac{\rho(\zeta)}{(\zeta-z)^2} + \frac{\overline{\zeta}^{\,2|n|}\overline{\rho(\zeta)}}{(1-z\overline{\zeta})^2}\right] dT_\zeta. \tag{11}$$

The operator $T_n^0\rho$ for $n \geq 0$ is completely continuous, and the norms of the operators $S_n^0\rho$ for $n \geq 0$ and $S_n\rho$ for $n < 0$ are equal to one if $\rho \in L_2(D)$ [1].

Theorem 1. *The solution of equation (7) is bounded in the norm in $L_p(D)$ by a number depending only on μ_0 and K.*

Proof. Suppose the contrary. Let there exist a sequence of coefficients $\mu_{1m}, \mu_{2m}, a_{1m}, a_{2m}, g_m$, satisfying conditions (3), (4), such that the sequence of equations

$$\rho_m + \mu_{1m}S_n\rho_m + \mu_{2m}\overline{S}_n\rho + a_{1m}T_n\rho_m + a_{2m}\overline{T}\rho_m = g_m \tag{12}$$

tends to infinity in the norm of $L_p(D)$.

Set $\tau_m = \rho_m/\|\rho_m\|$, $\|\tau_m\| = 1$. In view of the weak compactness of $\{\tau_m\}$, one can extract from it a weakly convergent subsequence. Let this be $\tau_m \to \tau_0$. Set $w_m = T_n\tau_m$.

Consider two cases: $n \geq 0$ and $n < 0$.

Case $n \geq 0$. For $w_n(z)$ one can obtain the representation

$$w_m(z) = e^{\omega_m(z) - p_m[\chi_m(z)]}\left\{-\frac{1}{\pi}\iint_D \left[\frac{f_m(t)e^{p_m(t)}}{t-\chi_m(z)} + \right.\right.$$

$$\left.\left. + \frac{\chi_m^{2m+1}(z)\overline{f_m(t)}\,\overline{e^{p_m(t)}}}{1-\chi_m(z)\overline{t}}\right] dT_t + \Phi_m[\chi_m(z)]\right\}, \tag{13}$$

where $\Phi_m(\zeta)$ is a polynomial of degree $2n$, whose coefficients are bounded by a number independent of m; $\omega_m(z)$ is a sequence of solutions of the problem

$$\frac{\partial \omega_m}{\partial \bar z}+\mu_m\frac{\partial \omega_m}{\partial z}+A_m=0, \qquad \operatorname{Im}\omega_m\big|_\Gamma=0; \qquad \int_\Gamma \omega_m\,ds=0; \qquad |\mu_m|\le\mu_0; \qquad \|A_m\|_L\le 2K;$$

$\chi_m(z)$ is a sequence of homeomorphisms of the Beltrami system

$$\frac{\partial \chi_m}{\partial \bar z}+\mu_m\frac{\partial \chi_m}{\partial z}=0,$$

mapping the disk onto itself; $p_m(\zeta)$ is a function analytic in D;

$$\operatorname{Im}p(\zeta)\big|_\Gamma=-n\arg\frac{\psi_m(\zeta)}{\zeta}\bigg|_\Gamma;$$

$\psi_m(\zeta)$ is the homeomorphism inverse to $\chi_m(z)$, satisfying the equation

$$\frac{\partial \psi_m}{\partial \bar\zeta}-\mu_m(\psi_m)\frac{\partial \overline{\psi_m}}{\partial \bar\zeta}=0;$$

$$f_m(\zeta)=\frac{g_m[|\psi_m(\zeta)|]}{\|\rho\|_{L_p}}\,\frac{1}{1-|\mu_m|^2}\,\frac{1}{\partial\chi_m/\partial z}; \qquad \|f_m(\zeta)\|_{L_{p_1}}\le\frac{K_1}{\|\rho_m\|_{L_p}}; \qquad p_1=\frac{p^2}{2(p-1)},$$

$2<p_1<p$; K_1 depends only on μ_0,K,p. In view of the fact that $\|\rho_m\|_{L_p}\to\infty$,

$$\|f_m\|_{L_{p_1}}\to 0. \tag{14}$$

From the results of B. V. Boyarskii (3) it follows that the sequences $\omega_m(z)$, $\chi_m(z)$, $\psi_m(z)$, $w_m(z)$, $p_m(\zeta)$ are bounded in norm in the space $W_p^{(1)}(D)$, $p>2$. From the complete continuity of the embedding operator $W_p^{(1)}(D)$, $p>2$, into $C(\bar D)$ (4), it follows that these sequences are compact in $C(\bar D)$. Let $\omega_0(z)$, $\chi_0(z)$, $\psi_0(\zeta)$, $w_0(z)$, $p_0(\zeta)$ be limit points of our sequences in the space $C(\bar D)$; let $\Phi_0(\zeta)$ be a limit point in $C(\bar D)$ of the sequence of polynomials $\Phi_m(\zeta)$. Taking into account (13), (14), and passing to the limit in (13), we obtain the function

$$w_0(z)=e^{\omega_0(z)-p_0[\chi_0(z)]}\Phi_0[\chi_0(z)]; \tag{15}$$

$\chi_0(z)$ is a homeomorphism, since $\chi_m(\psi_m(\zeta))=\zeta$. In view of uniform convergence, $w_0(z)$ satisfies condition (5). As follows from (1), $w_0(z)$ has on Γ not fewer than $2(n+1)$ zeros. But then from (15), in view of the fact that $\Phi_0(\zeta)$ is of full degree $2n$, it follows that $w_0(z)\equiv 0$.

Thus, from $w_m(z)$ we can extract a subsequence converging uniformly to zero. Let this be the sequence $w_m(z)$ itself. Then τ_m converges weakly to zero, $\|a_{im}T_n\tau_m\|_{L_p} \to 0$, $\|\mu_{im}T_n\tau_m\|_{L_p} \to 0$. But τ_m satisfies the equation

$$\tau_m + \mu_{1m}S_n^0\tau_m + \mu_{2m}\overline{S_n^0}\tau_m = \frac{g_m}{\|\rho_m\|} - a_{1m}T_n\tau_m -$$

$$- a_{2m}\overline{T}_n\tau_m - \mu_{1m}T_n^0\tau_m - \mu_{2m}\overline{T_n^0}\tau_m. \tag{16}$$

Choose p from the interval $2 < p < 2 + \varepsilon$ so that $\mu_0\|S_n^0\|_{L_p} < 1$, which can be done by virtue of the continuous dependence of $\|S_n^0\|_{L_p}$ on p and $\|S_n^0\|_{L_2} = 1$. From equation (16) one obtains the estimate

$$\|\tau_m\|_{L_p} \leq \frac{1}{1 - \mu_0\|S_n^0\|_{L_p}} \left\| \frac{g_m}{\|\rho_m\|} - a_{1m}T_n\tau_m - a_{2m}\overline{T}_n\tau_m - \right.$$

$$\left. - \mu_{1m}T_n^0\tau_m - \mu_{2m}\overline{T_n^0}\tau_m \right\|_{L_p} \to 0.$$

This contradicts the fact that $\|\tau_m\| = 1$. Consequently, for the case of nonnegative index the theorem is proved.

Case $n < 0$**.** The representation formula has the form

$$w_m(z) = e^{\omega_m(z) - p_m[\chi_m(z)]} \left\{ -\frac{1}{\pi} \iint_D \left[\frac{f_m(t)e^{p_m(t)}}{t - \chi_m(z)} + \frac{\bar{t}^{\,2|n|-1}\overline{f_m(t)}\, e^{\overline{p_m(t)}}}{1 - \bar{t}\chi_m(z)} \right] dT_t \right\},$$

and the remaining arguments are carried out analogously.

Theorem 2. The solutions of Problems 1 and 2 are bounded in norm in $W_p^{(1)}(D)$, $p > 2$, by a number depending only on μ_0 and K.

The proof follows immediately from Theorem 1 and the representation formulas

$$w = T_n\rho, \qquad \frac{\partial w}{\partial \bar{z}} = \rho, \qquad \frac{\partial w}{\partial z} = S_n\rho.$$

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Note: Figure translations are in progress. See original paper for figures.