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Abstract

Full Text

MATHEMATICAL PHYSICS

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THE MILNE PROBLEM WITH AN ARBITRARY INDICATRIX

(Presented by Academician M. V. Keldysh, 13 VIII 1957)

The Milne problem on the propagation of radiation in a plane-parallel scattering and absorbing medium filling the half-space $z > 0$ of three-dimensional space leads to the integral equation:

$$\psi(\tau, \mu) = \hat{A}\psi(\tau, \mu) + F(\tau, \mu), \quad (1)$$

$$\hat{A}f(\tau, \mu) = \int_0^\infty e^{-\rho} Y(\tau - \rho\mu) d\rho \int g_0(\vec{\Omega}\vec{\Omega}') f(\tau - \rho\mu, \mu') d\vec{\Omega}'. \quad (2)$$

Here τ is the optical thickness, measured from the boundary of the medium; $\vec{\Omega}$ is the unit vector of the direction of propagation of the radiation; $\mu = \vec{\Omega}\mathbf{k}$, $\mu' = \vec{\Omega}'\mathbf{k}$, where \mathbf{k} is the unit vector of the axis Oz .

$\psi(\tau, \mu)$ determines the phase density of radiation at the point (τ, μ) ; $2\pi g_0(\nu)$ is the probability density of scattering in one elementary act through the angle $\arccos \nu$, so that

$$1 - 2\pi \int_{-1}^1 g_0(\nu) d\nu$$

is the corresponding probability of absorption. By $Y(x)$ is denoted the Heaviside function: $Y(x) = 0$ for $x \leq 0$; $Y(x) = 1$ for $x > 0$. The free term $F(\tau, \mu)$ is determined by the density of radiation sources and by the boundary condition:

$$F(\tau, \mu) = \int_0^\infty e^{-\rho} Y(\tau - \rho\mu) \eta(\tau - \rho\mu, \mu) d\rho + Y(\mu) B(\mu) e^{-\tau/\mu}. \quad (3)$$

$\eta(\tau, \mu)$ is the phase density of the sources; $B(\mu)$ is the angular distribution of radiation incident from outside on the boundary of the medium.

If $g_0(\nu)$ is a polynomial in ν , then equation (1) can be reduced to an integral equation with respect to the function

$$\psi_0(\tau) = \int_{-1}^1 \psi(\tau, \mu) d\mu$$

with a kernel depending on the difference, and studied by the Hopf-Wiener method (for example, (1-4)). In the general case such a transition is impossible, and (1) with an arbitrary indicatrix, apparently, has not been studied. Moreover, in the general case there is no expression of $\psi(\tau, \mu)$ through $\psi_0(\tau)$. It is therefore expedient to study equation (1) directly.

Here the following assumptions on $g_0(\nu)$ are adopted:

- 1) $g_0(\nu) \geq 0$; $g_0(\nu) \in L_2(-1, 1)$; $2\pi \int_{-1}^1 g_0(\nu) d\nu = \vartheta \in (0, 1]$.
- 2) There exist $\alpha > 0$ and $[\mu_1, \mu_2] \subset (-1, 1)$ such that $g_0(\nu) \geq \alpha$ for almost all $\nu \in [\mu_1, \mu_2]$.

If we set

$$g(\mu, \mu') = \int_{\mu\mu' - \sqrt{1-\mu^2}\sqrt{1-\mu'^2}}^{\mu\mu' + \sqrt{1-\mu^2}\sqrt{1-\mu'^2}} g_0(\nu) (1 - \mu^2 - \mu'^2 - \nu^2 + 2\nu\mu\mu')^{1/2} d\nu, \quad (4)$$

then $g(\mu, \mu') \in L_2(\Delta)$, $\Delta = (-1, 1) \times (-1, 1)$, and

$$\hat{A}f(\tau, \mu) = \int_0^\infty e^{-\rho Y(\tau - \rho\mu)} d\rho \int_{-1}^1 g(\mu, \mu') f(\tau - \rho\mu, \mu') d\mu'.$$

From (4) one can derive the following important property of $g(\mu, \mu')$:

1. *If the interval $[-1, 1]$ is divided into two nonintersecting measurable parts E_1 and E_2 , and if almost everywhere on $E_1 \times E_2$ $g(\mu, \mu') = 0$, then either the measure of E_1 or the measure of E_2 is equal to zero.*

Equation (1) is closely connected with the following characteristic equation:

$$(1 + \lambda\mu)\varphi(\mu) = \hat{g}\varphi(\mu); \quad \hat{g}f(\mu) = \int_{-1}^1 g(\mu, \mu') f(\mu') d\mu'; \quad \lambda \in (-1, 1). \quad (5)$$

With the aid of 1), 2), and 1 one can prove that:

2. *Equation (5) admits a nontrivial nonnegative solution only for two values of the parameter λ : $\lambda = \pm\lambda_0$, $\lambda_0 \in [0, 1)$. To the value $\lambda = \lambda_0$ (respectively $\lambda = -\lambda_0$) there corresponds a unique solution of (5) in $L_2(-1, 1)$, $\varphi_{\lambda_0}(\mu)$ (respectively $\varphi_{-\lambda_0}(\mu)$). The solutions $\varphi_{\pm\lambda_0}(\mu)$ have the following properties:*

- a) $\|\varphi_{\pm\lambda_0}\| = 1$, $\varphi_{-\lambda_0}(\mu) = \varphi_{\lambda_0}(-\mu)$;
 b) $\varphi_{\lambda_0}(\mu)$ is continuous on $[-1, 1]$;

$$\inf_{\mu \in [-1, 1]} \varphi_{\lambda_0}(\mu) > 0;$$

- c) if $\vartheta = 1$, then $\lambda_0 = 0$ and $\varphi_{\lambda_0}(\mu) \equiv \text{const}$; if $\vartheta < 1$, then $\lambda_0 > 0$;
 d) for $\vartheta < 1$, $(\mu, \varphi_{\lambda_0}^2) < 0$ (x, y) denotes the scalar product in $L_2(-1, 1)$.

Let k be a complex variable. Denote by $\Gamma(a)$, where a is a real number, the cut in the k -plane:

$$\Gamma(a) = \{k \mid \text{Re } k = a, |k| \geq a\}.$$

Put $g_0^{(1)} = 2\pi(\mu, g_0)$.

Then:

3. *There exists $\lambda_1 \in (\lambda_0, 1]$ such that, whatever the bounded $\alpha(\mu) \in L_2(-1, 1)$, the solution $\varphi(k, \mu) \in L_2(-1, 1)$ of the equation*

$$(1 + k\mu)\varphi(k, \mu) = \hat{g}\varphi(k, \mu) + \alpha(\mu)$$

for $k \in \Gamma(\lambda_1)$, $k \neq \pm\lambda_0$, exists, is unique, and has the form

$$\varphi(k, \mu) = -\frac{3}{2k^2}(\alpha, 1)(1 - g_0^{(1)}) + 3 \frac{(\alpha, 1)\mu + (\alpha, \mu)}{2k} + \Phi_{\alpha, g_0}(k, \mu), \quad \text{if } \vartheta = 1;$$

$$\varphi(k, \mu) = \frac{(\alpha, \varphi_{\lambda_0}) \varphi_{\lambda_0}(\mu)}{(\mu, \varphi_{\lambda_0}^2) k - \lambda_0} + \frac{(\alpha, \varphi_{-\lambda_0}) \varphi_{-\lambda_0}(\mu)}{(\mu, \varphi_{-\lambda_0}^2) k + \lambda_0} + \Phi_{\alpha, g_0}(k, \mu), \quad \text{if } \vartheta < 1,$$

where $\Phi_{\alpha, g_0}(k, \mu)$ is determined by the functions $g_0(\mu)$ and $\alpha(\mu)$ and is an analytic function of k on the whole k -plane with the cut $\Gamma(\lambda_1)$. If F is a bounded closed set in the k -plane not containing the points $\pm\lambda_0$, $F \cap \Gamma(\lambda_1) = \emptyset$, then the solution $\varphi(k, \mu)$ is bounded on the set $F \times [-1, 1]$ of points (k, μ) .

Denote by H the set of all points (τ, μ) , $\tau \in (-\infty, \infty)$, $\mu \in [-1, 1]$, and by H^+ that part of H in which $\tau > 0$. Let $\mathfrak{M}, \mathfrak{M}^+$ be the sets of functions defined and measurable on H, H^+ , respectively, and let L, L^+ be the parts of $\mathfrak{M}, \mathfrak{M}^+$ consisting of nonnegative functions. Put

$$\mathfrak{M}(\delta_1, \delta_2) = \{\psi(\tau, \mu) \mid \psi \in \mathfrak{M}, |\psi(\tau, \mu)| \leq \text{const} \cdot Y(-\tau)e^{-\delta_1\tau} + \text{const} \cdot Y(\tau)e^{-\delta_2\tau}\},$$

$$\mathfrak{M}^+(\delta) = \{\psi(\tau, \mu) \mid \psi \in \mathfrak{M}^+, |\psi(\tau, \mu)| \leq \text{const} \cdot e^{-\delta\tau}\}.$$

Then $\hat{A}[L] \subset L$, $\hat{A}[L^+] \subset L^+$ and, if $\delta > -1$, $\delta_2 > -1$, δ_1 is arbitrary, then $\hat{A}[\mathfrak{M}(\delta_1, \delta_2)] \subset \mathfrak{M}(-1, \delta_2)$ and $\hat{A}[\mathfrak{M}^+(\delta)] \subset \mathfrak{M}^+(\delta)$. Moreover, \hat{A} carries every function from $\mathfrak{M}(\delta_1, \delta_2)$ (respectively, $\mathfrak{M}^+(\delta)$) that is continuous jointly in (τ, μ) into a function that is again continuous. If $\chi(\mu)$ is bounded, then, as is easy to compute, for $\sigma \in (-1, 1)$

$$\hat{A}e^{\sigma\tau}\chi(\mu) = e^{\sigma\tau} \frac{\hat{g}\chi(\mu)}{1 + \sigma\mu} - \frac{\hat{g}\chi(\mu)}{1 + \sigma\mu} Y(\mu)e^{-\tau/\mu}. \quad (6)$$

Using (6), item 2, and some other properties of \hat{g} , one can prove the following assertions:

4. If $\psi(\tau, \mu) \in \mathfrak{M}^+(\delta)$, $|\delta| < \lambda_0$, then the series $\sum_{\nu=0}^{\infty} \hat{A}^\nu \psi(\tau, \mu)$ converges absolutely and uniformly in every bounded part of H^+ to a function

$$\hat{S}\psi(\tau, \mu) = \sum_{\nu=0}^{\infty} \hat{A}^\nu \psi(\tau, \mu) \in \mathfrak{M}^+(\delta).$$

5. For every $\vartheta \in (0, 1]$, $\lim_{n \rightarrow \infty} \hat{A}^n 1 = 0$, $(\tau, \mu) \in H^+$. If $\vartheta = 1$, then the series

$$\sum_{\nu=0}^{\infty} \hat{A}^\nu Y(\mu)e^{-\tau/\mu}$$

converges to unity uniformly in every bounded closed part of H^+ .

Hence:

6. If in (3) $B(\mu)$ is bounded, $\eta(\tau, \mu) \in \mathfrak{M}^+(\delta)$, $\delta > -\lambda_0$, then (1) has an everywhere finite solution $\psi(\tau, \mu)$, representable by the Neumann series: $\psi(\tau, \mu) = \hat{S}F(\tau, \mu)$. If $|\delta| < \lambda_0$, then $\hat{S}F(\tau, \mu) \in \mathfrak{M}^+(\delta)$.

7. For $\vartheta \in (0, 1]$,

$$\hat{S}\varphi_{\lambda_0}(-\mu)Y(\mu)e^{-\tau/\mu} = \varphi_{\lambda_0}(-\mu)e^{-\lambda_0\tau}, \quad (\tau, \mu) \in H^+.$$

With the aid of item 6 it is not difficult to find a nontrivial solution of the homogeneous equation (1):

8. If $\vartheta \in (0, 1)$ (respectively, $\vartheta = 1$), then the function

$$\psi_{g_0}(\tau, \mu) = \varphi_{\lambda_0}(\mu)e^{\lambda_0\tau} - \hat{S}\varphi_{\lambda_0}(\mu)Y(\mu)e^{-\tau/\mu}$$

$$\left(\text{respectively, } \bar{\psi}_{g_0}(\tau, \mu) = \tau + \frac{1-\mu}{1-g_0^{(1)}} - \hat{S} \frac{1-\mu}{1-g_0^{(1)}} Y(\mu)e^{-\tau/\mu} \right)$$

is a continuous, nonnegative, nontrivial solution on H^+ of the homogeneous equation (1); $\psi_{g_0} \in \mathfrak{M}^+(-\lambda_0)$; $\bar{\psi}_{g_0} \in \mathfrak{M}^+(-\varepsilon)$, $\varepsilon > 0$.

9. An everywhere finite function $\psi(\tau, \mu)$ is not a solution of the homogeneous equation (1) if there exist $\tau_1 \geq 0$, $A_0 > 0$, $\lambda > \lambda_0$ such that $\psi(\tau, \mu) \geq A_0 e^{\lambda\tau}$ for $\tau > \tau_1$. The homogeneous equation (1) has no nontrivial solutions in the classes $\mathfrak{M}^+(\delta)$, $\delta > -\lambda_0$ for $\vartheta < 1$, and $\delta \geq 0$ for $\vartheta = 1$.

Let in (3) $\eta(\tau, \mu) = 0$, and let $B(\mu)$ be bounded. Then (1) has a unique bounded solution $\psi(\tau, \mu) = \hat{S}B(\mu)Y(\mu)e^{-\tau/\mu}$. From (1) it is not difficult to derive that there exists $\lim_{\tau \rightarrow 0-0} \psi(\tau, \mu) = \psi(0, \mu)$. Put, for $\tau < 0$, $\psi(\tau, \mu) = 0$, $F(\tau, \mu) = -\hat{A}\psi(\tau, \mu)$. Then $\psi(\tau, \mu) \in \mathfrak{M}(-\infty, \lambda_0)$, $F(\tau, \mu) \in \mathfrak{M}(-1, 1)$, and $\psi(\tau, \mu)$ satisfies (1) for all $(\tau, \mu) \in H$. Let

$$\tilde{\psi}(k, \mu) = \int_{-\infty}^{\infty} e^{k\tau} \psi(\tau, \mu) d\tau.$$

From (1) we find that

$$(1 - k\mu)\tilde{\psi}(k, \mu) = \hat{g}\tilde{\psi}(k, \mu) + \mu\psi(0, \mu), \quad \operatorname{Re} k \in (-1, \lambda_0). \quad (7)$$

$\tilde{\psi}(k, \mu)$ is analytic in k for $\operatorname{Re} k < \lambda_0$. With the aid of item 3, $\tilde{\psi}(k, \mu)$ can be analytically continued to the region of the k -plane mentioned in item 3.

On the other hand, from (7) it follows that:

10. Uniformly with respect to all $\mu \in [-1, 1]$ and k satisfying the condition $\left| \frac{\operatorname{Im} k}{\operatorname{Re} k} \right| \geq \text{const} > 0$,

$$\lim_{|k| \rightarrow \infty} \tilde{\psi}(k, \mu) = 0.$$

Hence the applicability of Jordan's lemma to the integral follows:

$$\psi(\tau, \mu) = \frac{1}{2\pi i} \int_{k_1 - i\infty}^{k_1 + i\infty} \tilde{\psi}(k, \mu) e^{-k\tau} dk, \quad k_1 \in (-1, \lambda_0).$$

Deforming and shifting the contour of integration in the proper way, with the aid of the formulas of item 3 we find:

11. If $\vartheta \in (0, 1)$ (respectively, $\vartheta = 1$), then in the problem under consideration

$$\psi(\tau, \mu) = \frac{(\mu\varphi_{-\lambda_0}(\mu), \psi(0, \mu))}{(\mu, \varphi_{-\lambda_0}^2)} e^{-\lambda_0\tau} \varphi_{-\lambda_0}(\mu) + o(e^{-\lambda_2\tau}), \quad \lambda_2 \in (\lambda_0, \lambda_1)$$

$$\text{(respectively, } \psi(\tau, \mu) = \frac{3}{2}(\mu^2, \psi(0, \mu)) + o(e^{-\lambda_2\tau}), \quad \lambda_2 \in (0, \lambda_1)).$$

In addition, from (7) it is not difficult to obtain theorems on the vanishing of the total flux, on the constancy of the K -integral, and their generalizations to the nonconservative case (for example, $(\mu\varphi_{\lambda_0}(\mu), \psi(\tau, \mu)) = 0$ for $\tau > 0$).

The arguments carried out are also applicable to the homogeneous equation (1). Here the following results are obtained:

12. If $\psi(\tau, \mu) \geq 0$ is a nontrivial solution of the homogeneous equation (1) and $\psi(\tau, \mu) \in \mathfrak{M}^+(\sigma)$, $\sigma > -1$, then for $\vartheta \in (0, 1)$ (respectively, $\vartheta = 1$) $\psi(\tau, \mu)$, up to normalization, coincides with $\psi_{g_0}(\tau, \mu)$ (respectively, with $\bar{\psi}_{g_0}(\tau, \mu)$).

Assertions 9 and 12 form a uniqueness theorem for the solution of the homogeneous equation (1).

$$\psi_{g_0}(\tau, \mu) = \varphi_{\lambda_0}(\mu) e^{\lambda_0\tau} - \frac{(\mu\varphi_{-\lambda_0}(\mu), \psi_{g_0}(0, \mu))}{(\mu\varphi_{\lambda_0}(\mu), \psi_{g_0}(0, \mu))} e^{-\lambda_0\tau} \varphi_{-\lambda_0}(\mu) + o(e^{-\lambda_2\tau}),$$

$$\lambda_2 \in (\lambda_0, \lambda_1);$$

$$\bar{\psi}_{g_0}(\tau, \mu) = \tau - \frac{\mu}{1 - g_0^{(1)}} + \frac{3}{2}(\mu^2, \bar{\psi}_{g_0}(0, \mu)) + o(e^{-\lambda_2\tau}), \quad \lambda_2 \in (0, \lambda_1).$$

From this it is easy to obtain certain other relations (for example, formulas for the spatial distribution of radiation, extrapolated lengths, etc.).

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