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Abstract

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ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER, WHOSE SYSTEMS OF "FAST MOTIONS" ARE CLOSE TO HAMILTONIAN ONES

(Presented by Academician P. S. Aleksandrov, 18 IV 1958)

We consider the system of differential equations

$$\begin{aligned} \varepsilon\frac{dx}{dt} &= \frac{\partial H(x,y,z_1,\dots,z_l)}{\partial y} + \varepsilon X(x,y,z_1,\dots,z_l,\varepsilon), \\ \varepsilon\frac{dy}{dt} &= -\frac{\partial H(x,y,z_1,\dots,z_l)}{\partial x} + \varepsilon Y(x,y,z_1,\dots,z_l,\varepsilon), \\ \frac{dz_j}{dt} &= Z_j(x,y,z_1,\dots,z_l,\varepsilon) \qquad (j=1,2,\dots,l), \end{aligned} \tag{1}$$

where ε is a small positive parameter.

The problem of investigating this system and deriving from it the known results of V. M. Volosov ([1–5]) was posed by L. S. Pontryagin in his report at V. I. Smirnov' s seminar in Leningrad in April 1957.

In the fast time $\tau = t/\varepsilon$, system (1) has the form

$$\begin{aligned} \frac{dx}{d\tau} &= \frac{\partial H(x,y,z_1,\dots,z_l)}{\partial y} + \varepsilon X(x,y,z_1,\dots,z_l,\varepsilon), \\ \frac{dy}{d\tau} &= -\frac{\partial H(x,y,z_1,\dots,z_l)}{\partial x} + \varepsilon Y(x,y,z_1,\dots,z_l,\varepsilon), \\ \frac{dz_j}{d\tau} &= \varepsilon Z_j(x,y,z_1,\dots,z_l,\varepsilon) \qquad (j=1,2,\dots,l). \end{aligned} \tag{2}$$

For $\varepsilon = 0$, system (2) becomes the Hamiltonian system

$$\begin{aligned} \frac{dx}{d\tau} &= \frac{\partial H(x, y, z_1, \dots, z_l)}{\partial y}, \\ \frac{dy}{d\tau} &= -\frac{\partial H(x, y, z_1, \dots, z_l)}{\partial x}, \\ \frac{dz_j}{d\tau} &= 0 \qquad (j = 1, \dots, l). \end{aligned} \tag{3}$$

The first integral

$$H(x, y, z_1, \dots, z_l) = h \tag{4}$$

represents the family of all phase trajectories of system (3) on each plane $z_j =$ const $(j = 1, \dots, l)$ of the Euclidean space E_{2+l} of the variables $x, y, z_1, \dots, z_l$.

Take some point $(x^0, y^0, z_1^0, \dots, z_l^0)$ in E_{2+l} that is not an equilibrium position of system (3). $h^0 = H(x^0, y^0, z_1^0, \dots, z_l^0)$, $z_1^0, \dots, z_l^0$

determine only one phase trajectory of system (3), $H(x, y, z_1^0, \dots, z_l^0) = h^0$, passing through $(x^0, y^0, z_1^0, \dots, z_l^0)$. Let this phase trajectory be closed. Then in the space E_{2+l} there exists a certain neighborhood G of this trajectory in which: 1) the phase trajectories of system (3) are closed; 2) each complex $(h, z_1, \dots, z_l)$ from the neighborhood of $(h^0, z_1^0, \dots, z_l^0)$ determines only one phase trajectory of system (3); 3) on each phase trajectory (4) of system (3) one can choose one initial point $\{\alpha(h, z_1, \dots, z_l), \beta(h, z_1, \dots, z_l)\}$, for example as the intersection of the phase trajectories (4) with some smooth curve passing through $\{x^0, y^0\}$ and not tangent to the phase trajectory $H(x, y, z_1, \dots, z_l) = H(x^0, y^0, z_1, \dots, z_l)$ at the point $(x^0, y^0, z_1, \dots, z_l)$.

We investigate the solution of system (1) $x(t, \varepsilon), y(t, \varepsilon), z_1(t, \varepsilon), \dots, z_l(t, \varepsilon)$, passing through an arbitrary point $(x_0, y_0, z_{10}, \dots, z_{l0})$ of G at $t = t_0$.

Theorem. *Let the functions $H(x, y, z_1, \dots, z_l)$, $\partial H(x, y, z_1, \dots, z_l)/\partial x$, $\partial H(x, y, z_1, \dots, z_l)/\partial y$ be defined and continuous in G together with the partial derivatives with respect to all variables up to the second order inclusive, and let the functions $X(x, y, z_1, \dots, z_l, \varepsilon)$, $Y(x, y, z_1, \dots, z_l, \varepsilon)$, $Z_j(x, y, z_1, \dots, z_l, \varepsilon)$ $(j = 1, 2, \dots, l)$ be continuous in G together with partial derivatives up to the first order inclusive and differentiable with respect to ε for $\varepsilon \geqslant 0$.*

Then there exists such an ε_0 that, for any $0 < \varepsilon \leqslant \varepsilon_0$:

1) The functions $h(t, \varepsilon) = H[x(t, \varepsilon), y(t, \varepsilon), z_1(t, \varepsilon), \dots, z_l(t, \varepsilon)]$, $z_1(t, \varepsilon), \dots, z_l(t, \varepsilon)$ on $[t_0, L]$ coincide, to within quantities of order $o(\varepsilon)$, with the solution $\bar{h}(t), \bar{z}_1(t), \dots, \bar{z}_l(t)$ of the autonomous system of ordinary differential equations, independent of ε:

$$\frac{dh}{dt} = A(h, z_1, \dots, z_l),$$

$$\frac{dz_j}{dt} = A_j(h, z_1, \dots, z_l) \qquad (j = 1, \dots, l), \tag{5}$$

passing at $t = t_0$ through $h_0 = H(x_0, y_0, z_{10}, \dots, z_{l0}), z_{10}, \dots, z_{l0}$ and remaining in G together with some ρ-neighborhood on the finite interval $[t_0, L]$. In (5) the functions $A(h, z_1, \dots, z_l)$, $A_j(h, z_1, \dots, z_l)$ $(j = 1, \dots, l)$ are expressed in terms of the functions of the right-hand sides of system (1) by the formulas

$$A = \frac{1}{T} \oint_{H=h} \left[X \frac{\partial H}{\partial x} + Y \frac{\partial H}{\partial y} + \sum_{i=1}^{l} Z_i \frac{\partial H}{\partial z_i} \right] \left[\left(\frac{\partial H}{\partial x} \right)^2 + \left(\frac{\partial H}{\partial y} \right)^2 \right]^{-1/2} ds,$$

$$A_j = \frac{1}{T} \oint_{H=h} Z_j \left[\left(\frac{\partial H}{\partial x} \right)^2 + \left(\frac{\partial H}{\partial y} \right)^2 \right]^{-1/2} ds \qquad (j = 1, \dots, l), \tag{6}$$

where $X = X(x, y, z_1, \dots, z_l, 0)$; $Y = Y(x, y, z_1, \dots, z_l, 0)$; $Z_j = Z_j(x, y, z_1, \dots, z_l, 0)$; $H = H(x, y, z_1, \dots, z_l)$; $T = T(h, z_1, \dots, z_l) =$

$$= \oint_{H=h} \left[\left(\frac{\partial H}{\partial x} \right)^2 + \left(\frac{\partial H}{\partial y} \right)^2 \right]^{-1/2} ds;$$

s is the arc length of the phase trajectory (4);

$$H(x, y, z_1, \dots, z_l) = h \quad (h, z_1, \dots, z_l \text{ fixed}).$$

2) The functions $x(t, \varepsilon), y(t, \varepsilon)$, to within quantities of order $o(\varepsilon)$, coincide with the functions

$$x(t, \varepsilon) = \tag{7}$$

$$= x^* \left(\varphi(t_0) + \frac{1}{\varepsilon} \int_{t_0}^{t} \frac{1}{T[\bar{h}(r), \bar{z}_1(r), \dots, \bar{z}_l(r)]} \, dr + \nu(t, \varepsilon), \bar{h}(t), \bar{z}_1(t), \dots, \bar{z}_l(t) \right),$$

$$y(t, \varepsilon) = \tag{8}$$

$$= y^* \left(\varphi(t_0) + \frac{1}{\varepsilon} \int_{t_0}^{t} \frac{1}{T[\bar{h}(r), \bar{z}_1(r), \dots, \bar{z}_l(r)]} \, dr + \nu(t, \varepsilon), \bar{h}(t), \bar{z}_1(t), \dots, \bar{z}_l(t) \right),$$

and $x^*(\varphi, h, z_1, \dots, z_l)$ and $y^*(\varphi, h, z_1, \dots, z_l)$ are periodic in φ with period 1, and

$$x^*(\varphi, h, z_1, \dots, z_l) \equiv \tilde{x}(T\varphi, h, z_1, \dots, z_l) \equiv \tilde{x}(\tau, h, z_1, \dots, z_l),$$

$$y^*(\varphi, h, z_1, \dots, z_l) \equiv \tilde{y}(T\varphi, h, z_1, \dots, z_l) \equiv \tilde{y}(\tau, h, z_1, \dots, z_l);$$

$\tilde{x}(\tau, h, z_1, \dots, z_l)$, $\tilde{y}(\tau, h, z_1, \dots, z_l)$ is a solution of system (3) passing through $\{\alpha(h, z_1, \dots, z_l), \beta(h, z_1, \dots, z_l)\}$ at $\tau = 0$; the period of this solution is $T(h, z_1, \dots, z_l)$; $\varphi_0 = \varphi(t_0)$ is found from $x_0 = x^*(\varphi_0, h_0, z_{10}, \dots, z_{l0})$, $y_0 = y^*(\varphi_0, h_0, z_{10}, \dots, z_{l0})$ (the function $\nu(t, \varepsilon) = \varphi(t, \varepsilon) - \bar{\varphi}(t, \varepsilon)$ will be discussed below).

Proof. The change of variables x, y to φ, h by the formula

$$x = x^*(\varphi, h, z_1, \dots, z_l), \qquad y = y^*(\varphi, h, z_1, \dots, z_l) \tag{9}$$

transforms system (1) into the system

$$\frac{dh}{dt} = \mathfrak{A}(\varphi, h, z_1, \dots, z_l, \varepsilon),$$

$$\frac{dz_j}{dt} = \mathfrak{A}_j(\varphi, h, z_1, \dots, z_l, \varepsilon), \tag{10}$$

$$\frac{d\varphi}{dt} = \frac{1}{\varepsilon T(h, z_1, \dots, z_l)} +$$

$$+ \frac{1}{T(h, z_1, \dots, z_l)} \left[X \frac{\partial y^*}{\partial h} - Y \frac{\partial x^*}{\partial h} + \sum_{i=1}^{l} Z_i \left(\frac{\partial x^*}{\partial h} \frac{\partial y^*}{\partial z_i} - \frac{\partial y^*}{\partial h} \frac{\partial x^*}{\partial z_i} \right) \right],$$

where

$$\mathfrak{A} \equiv X \frac{\partial H}{\partial x} + Y \frac{\partial H}{\partial y} + \sum_{i=1}^{l} Z_i \frac{\partial H}{\partial z_i}, \qquad \mathfrak{A}_j \equiv Z_j,$$

$$X = X[x^*(\varphi, h, z_1, \dots, z_l),\ y^*(\varphi, h, z_1, \dots, z_l),\ z_1, \dots, z_l,\ \varepsilon],$$

$$Y = Y[x^*(\varphi, h, z_1, \dots, z_l),\ y^*(\varphi, h, z_1, \dots, z_l),\ z_1, \dots, z_l,\ \varepsilon],$$

$$Z_j = Z_j[x^*(\varphi, h, z_1, \dots, z_l),\ y^*(\varphi, h, z_1, \dots, z_l),\ z_1, \dots, z_l,\ \varepsilon].$$

But the solution of system (10) $h(t,\varepsilon), z_1(t,\varepsilon), \dots, z_l(t,\varepsilon), \varphi(t,\varepsilon)$, passing through $h_0, z_{10}, \dots, z_{l0}, \varphi_0$ at $t = t_0$, is related to the solution $\bar{h}(t)$, $\bar{z}_1(t), \dots, \bar{z}_l(t), \bar{\varphi}(t,\varepsilon)$ of the averaged system

$$\frac{dh}{dt} = \int_0^1 \mathfrak{A}(\varphi, h, z_1, \dots, z_l, 0)\, d\varphi \equiv A(h, z_1, \dots, z_l),$$

$$\frac{dz_j}{dt} = \int_0^1 \mathfrak{A}_j(\varphi, h, z_1, \dots, z_l, 0)\, d\varphi \equiv A_j(h, z_1, \dots, z_l) \qquad (j = 1, \dots, l,$$

$$\frac{d\varphi}{dt} = \frac{1}{\varepsilon T(h, z_1, \dots, z_l)}, \tag{11}$$

passing through the same initial point $h_0, z_{10}, \dots, z_{l0}, \varphi_0$, as follows:

$$\begin{aligned} |h(t,\varepsilon) - \bar{h}(t)| &\le o(\varepsilon), \\ |z_j(t,\varepsilon) - \bar{z}_j(t)| &\le o(\varepsilon), \\ |\varphi(t,\varepsilon) - \bar{\varphi}(t,\varepsilon)| &\le o(1) \qquad (o(\varepsilon) > 1,\ o(1) > 0). \end{aligned} \tag{12}$$

Relations (12) and (9) prove the theorem.

Remark. It is also proved that if the point $(x_0,\ y_0,\ z_{10}, \dots,\ z_{l0})$ of E_{2+l} is a singular point of system (3) of the type of a nondegenerate center, then there exists some neighborhood G_0 of this point such that, under the hypotheses of the theorem, the solution of system (1) $x(t,\varepsilon), y(t,\varepsilon), z_1(t,\varepsilon), \dots,\ z_l(t,\varepsilon)$, passing through $x_{00}, y_{00}, z_{10}, \dots,\ z_{l0}$ at $t = t_0$, coincides on the interval $[t_0, L]$, up to an error of order $o(\varepsilon)$, with the solution of the degenerate system

$$\frac{\partial H(x, y, z_1, \dots, z_l)}{\partial y} = 0 \qquad \left(|x_0 - x_{00}| + |y_0 - y_{00}| \le o(\varepsilon)\right),$$

$$\frac{\partial H(x, y, z_1, \dots, z_l)}{\partial x} = 0, \tag{13}$$

$$\frac{dz_j}{dt} = Z_j(x, y, z_1, \dots, z_l, 0) \qquad (j = 1, \dots, l),$$

passing through $x_0, y_0, z_{10}, \dots, z_{l0}$ at $t = t_0$ and remaining in G_0 together with some ρ_0-neighborhood on $[t_0, L]$.

Let us note that the equation

$$\mu u^{(n)} + Q(t, u, u', \dots, u^{(n-2)}) = 0, \tag{14}$$

where μ is a small positive parameter; $Q(t, u, \ldots, u^{(n-2)}) = 0$ has the root $u^{(n-2)} = f(t, u, \ldots, u^{(n-s)})$, $\operatorname{sign} Q = \operatorname{sign}(u^{(n-2)} - f)$; $m|u^{(n-2)} - f| \leq |Q| \leq M|u^{(n-2)} - f|$; m, M are certain positive constants, considered in (1-4), is reduced by the substitution

$$t = z_1, \qquad u = z_2, \ldots, \qquad u^{(n-3)} = z_{n-1}, \qquad u^{(n-2)} = x, \qquad \varepsilon = \sqrt{\mu}, \qquad \varepsilon \frac{dx}{dt} = y,$$

$$H(x, y, z_1, \ldots, z_{n-1}) = \frac{y^2}{2} + \int_0^x Q(z_1, \ldots, z_{n-1}, p)\, dp \tag{15}$$

to the system

$$\varepsilon \frac{dx}{dt} = \frac{\partial H(x, y, z_1, \ldots, z_{n-1})}{\partial y},$$

$$\varepsilon \frac{dy}{dt} = -\frac{\partial H(x, y, z_1, \ldots, z_{n-1})}{\partial x},$$

$$\frac{dz_1}{dt} = 1,$$

$$\frac{dz_i}{dt} = z_{i+1} \qquad (i = 2, \ldots, n-2),$$

$$\frac{dz_{n-1}}{dt} = x, \tag{16}$$

where the corresponding Hamiltonian system

$$\varepsilon \frac{dx}{dt} = y,$$

$$\varepsilon \frac{dy}{dt} = -Q(z_1, \ldots, z_{n-1}, x)$$

has the singular point $x = f(z_1, \ldots, z_{n-1}),\ y = 0$ of the type of a nondegenerate center. System (16), however, is a special case of system (1), and the results of papers (1-5) follow from the theorem obtained in the present article.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.