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**Abstract**

**Full Text**

## **Reports of the Academy of Sciences of the USSR**

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**MATHEMATICS**

**B. V. BOYARSKII**

### **A GENERAL REPRESENTATION OF SOLUTIONS OF AN ELLIPTIC SYSTEM OF $2n$ EQUATIONS IN THE PLANE**

*(Presented by Academician I. N. Vekua on 13 V 1958)*

1. In the present work we consider systems of first-order equations of the form

$$u_x = \tilde{A}u_y + \tilde{B}u + f, \quad (*)$$

where  $u$  is an unknown  $2n$ -component real vector;  $\tilde{A}$  and  $\tilde{B}$  are square matrices of order  $2n$ ,  $f$  is a real vector, given in some plane domain  $T$  of the variable  $z = x + iy$ . The ellipticity of the system (\*) means that at every point of the domain  $T$  the equation

$$\det(\tilde{A} - \lambda E) = 0$$

has no real roots <sup>(1)</sup>. We shall consider the system (\*) in the following complex canonical form:

$$w_{\bar{z}} - Qw_z = Aw + B\bar{w} + F, \quad (1)$$

where  $w(z)$  is an  $n$ -component complex unknown vector;  $A$  and  $B$  are complex square matrices of order  $n$ ;  $F$  is a complex vector;  $Q$  is a quasidiagonal matrix of the form  $Q = \{Q_1, Q_2, \dots, Q_p\}$ , where  $Q_i = \{q_{lk,i}\}$  are square matrices,  $q_{lk,i} = 0$  for  $k > l$ ,  $q_{ll,i} = q_i$ , and  $q_{lk,i} = \beta_i^{l-k}$  for  $k < l$ ,  $q_i$  and  $\beta_i$  being complex functions defined in  $T$ .

Equation (1) is the complex notation for the canonical form of the system (\*) in a real domain <sup>(1,4,5)</sup>. If  $T$  is simply connected, the elementary divisors of the matrix  $\tilde{A} - \lambda E$  preserve their multiplicity in  $T$ , the rank of each of the matrices  $\tilde{A} - \lambda_i E$ , where  $\det(\tilde{A} - \lambda_i E) = 0$ , is constant in  $T$ , and, in addition, the matrix

$\tilde{A}$  is continuously differentiable in  $T$ , then the system (\*) can be reduced to the form (1) in the entire domain  $T$  <sup>(1,4)</sup>.

By ellipticity, one has  $|q_i| < 1$  in  $T$ . If the system (\*) is assumed to be uniformly elliptic, i.e.  $|\operatorname{Im} \lambda_i| > \tilde{\lambda} > 0$  and  $|\lambda_i| \leq M$  ( $\tilde{\lambda}$  and  $M$  are positive constants), whenever  $\det(\tilde{A} - \lambda_{iE}) = 0$ , then  $|q_i| \leq q_0 < 1$ ,  $i = 1, \dots, p$ ,  $q_0 = \text{const}$ ; then it can be shown that in (1), without loss of generality, one may assume  $|\beta_i| < \beta_0$ ,  $i = 1, \dots, p$ , where  $\beta_0$  is a prescribed arbitrarily small positive number; for sufficiently small  $\beta_0$  one has

$$\max_i \sum_k |q_{ik}| \leq q'_0 < 1, \quad q'_0 = \text{const}. \quad (2)$$

In what follows we shall consider systems of the form (1) under the following assumptions: 1) the matrices  $A$ ,  $B$ , and  $Q$  are given on the whole plane, with  $A = B = Q = 0$  outside some sufficiently large circle  $K$ ; 2)  $Q$  possesses (generalized) derivatives  $Q_z$  and  $Q_{\bar{z}} \in L_p(K)$ ,  $p > 2$ ;  $A$  and  $B$  are measurable and bounded for  $z \in K$ . We note that, using

inequality (2), a number of problems for the system (1) can be solved under  $A, B \in L_p(K)$  and any measurable  $Q$  satisfying (2).

In the paper a theory of systems (1) is constructed, parallel to the theory of I. N. Vekua <sup>(2)</sup>, who studied the case  $n = 1$  and  $Q \equiv 0$ . The case  $n > 1$  differs from the case  $n = 1$  in a number of respects; all of them are connected with the invalidity, in the general case, of Liouville's theorem for systems (1) ( $n > 1$ ).

By a **regular solution** of the system (1) we shall mean a vector  $w(z)$  possessing generalized derivatives  $w_z, w_{\bar{z}} \in L_p$ ,  $p \geq 2$ , and satisfying equation (1) almost everywhere. An example is constructed of a system (1) which has a solution regular in the whole plane, vanishing at infinity and not identically equal to zero. However, in a number of cases Liouville's theorem for systems (1) remains valid. This holds, for example, in the case when  $Q, A, B$  are constant inside  $K$  and are equal to zero outside  $K$ .

## 2. Solutions of the system

$$\Phi_{\bar{z}} - Q\Phi_z = 0 \quad (3)$$

will be called  **$Q$ -holomorphic vectors**. The theory of a  $Q$ -holomorphic vector is completely analogous to the theory of a holomorphic vector. Under stronger assumptions than ours, a number of theorems of this theory are available in the work of Douglas <sup>(4)</sup>. For a  $Q$ -holomorphic vector the generalized Liouville theorem is valid.

**Theorem 1.** There exists a matrix  $V(z, t)$ , defined for all  $z$  and  $t$  and possessing the following properties:

1)

$$V'_z - (Q'V')_z = 0;$$

2)

$$V_t - QV_t = 0 \quad \text{for } z \neq t;$$

3)

$$\lim_{t \rightarrow \infty} tV(z, t) = \hat{V}(z); \quad \det \hat{V}(z) \neq 0 \text{ for no } z, \text{ including } z = \infty;$$

4)

$$\lim_{z \rightarrow \infty} zV(z, t) = E \quad \text{for all } t;$$

5)

$$\lim_{\delta \rightarrow 0} \int_{|z-t|=\delta} V(t, z) (dt + Q\overline{dt}) w(t) = -2\pi i w(z)$$

for any finite  $z$  and for any continuous vector  $w(z)$ .

Conditions 1), 2), 3), 4), 5) determine the matrix  $V(z, t)$  uniquely.

For the matrix  $V(z, t)$  the following expression can be indicated: denote by  $\zeta(z)$  the solution of equation (3) of the form

$$\zeta(z) = zE - \frac{1}{\pi} \iint_K \frac{\omega(t)}{t-z} dK_t.$$

Using (2), it is not difficult to show that  $\zeta(z)$  exists and is unique. Further, for  $z \neq t$  there exists the matrix  $(\zeta(t) - \zeta(z))^{-1}$ . To determine  $V(z, t)$  we put

$$V(z, t) = -(\zeta(t) - \zeta(z))^{-1} \varphi_z(z).$$

The matrix  $V(z, t)$  is naturally called the **Cauchy kernel** of equation (3). By  $\hat{V}(z, t)$  we shall denote the Cauchy kernel of the system

$$-\varphi_z + (Q'\varphi)_z = 0,$$

conjugate to (3). The kernels  $V$  and  $\hat{V}$  are connected by the relation

$$\hat{V}(z, t) = -V'(t, z).$$

The matrix  $V(z, t)$  plays the role of the matrix  $(t-z)^{-1}E$  in the theory of a holomorphic vector.

For any vector  $w(z)$ , continuous in a plane domain  $T$  bounded by a piecewise smooth contour  $L = L_0 + L_1 + \dots + L_m$  and belonging to the Sobolev class  $W_p^1(T)$ ,  $p > 1$ , the following generalization of the Pompeiu decomposition <sup>(2,3)</sup>, important in the theory of I. N. Vekua <sup>(2)</sup>, holds:

$$W(z) = \frac{1}{2\pi i} \int_L V(t, z)(dt + Q\bar{d}t) w(t) - \frac{1}{\pi} \iint_T V(t, z) \omega(t) dT_t, \quad (4)$$

where  $\omega(z) = w_{\bar{z}} - Qw_z$ .

By  $\overset{\circ}{V}(z)$  we shall denote the matrix solution of the equation

$$\overset{\circ}{V}_z - (Q'\overset{\circ}{V})_z = 0 \quad \text{of the form} \quad \overset{\circ}{V}(z) = E - \frac{1}{\pi} \iint_K \frac{\omega(t)}{t-z} dK_t.$$

Such a solution always exists. With its aid one can define the  $Q$ -derivative  $\partial_z^Q w$  and the  $Q$ -integral  $(Q) \int w$  of the vector  $w(z)$  by the formulas

$$\partial_z^Q w = \overset{\circ}{V}^{-1} w_z, \quad (Q) \int w = \int_{z_0}^z \overset{\circ}{V}(z)(dz + Q d\bar{z})w(z).$$

It is not difficult to verify that the  $Q$ -derivative and the  $Q$ -integral of a  $Q$ -holomorphic vector are again a  $Q$ -holomorphic vector <sup>(4)</sup>. The following lemma is important for applications to boundary-value problems:

**Lemma 1.** *Every  $Q$ -holomorphic vector  $w(z)$ , Hölder-continuous in the closed domain  $T + L$  ( $L$  is a smooth contour), admits the representation*

$$w(z) = \frac{1}{\pi i} \int_L V(t, z)(\bar{d}t + Q d\bar{t})\mu(t) + iC,$$

where  $\mu(t)$  is a real vector, Hölder-continuous, and  $C$  is a real constant vector; the vector  $C$  is determined uniquely by the vector  $w(z)$ ; the vector  $\mu(t)$  is determined up to an additive constant on each of the contours  $L_j$ ,  $j \geq 1$ , bounding the domain  $T$  and lying inside the contour  $L_0$ .

3. Formula (4) makes it possible to associate with system (1) the following system of Fredholm-type integral equations:

$$w(z) + \frac{1}{\pi} \iint_K V(t, z)(Aw + B\bar{w}) dK_t = \Phi(z). \quad (5)$$

Equation (5) is equivalent to system (1), if  $\Phi$  is a  $Q$ -holomorphic vector. If  $w(z)$  is given, then the corresponding  $\Phi(z)$  is found by the formula

$$\Phi(z) = \frac{1}{2\pi i} \int_L V(t, z)(dt + Q d\bar{t})w(t).$$

For  $n = 1$  and  $Q = 0$  system (5) becomes the system of integral equations placed by I. N. Vekua at the foundation of his theory <sup>(2)</sup>. As follows from the example

in §1, the homogeneous equation (5) may, for  $n > 1$ , admit a finite number of nontrivial solutions  $w_k(z)$ ,  $k = 1, 2, \dots, N$ . Therefore the nonhomogeneous equation (5) is solvable only when a finite number of equalities of the form

$$\operatorname{Re} \iint_K \Phi v_j dK = 0, \quad j = 1, 2, \dots, N,$$

are satisfied, where  $v_j$  are solutions of the homogeneous system of integral equations conjugate to (5) with respect to the scalar product

$$[\Phi, v] = \operatorname{Re} \iint_K (\Phi, \bar{V}) dK, \quad (\Phi, \psi) = \sum_{i=1}^n \Phi_i \psi_i.$$

By  $\psi_k(z)$ ,  $k = 1, 2, \dots, N$ , we shall everywhere denote regular nontrivial solutions vanishing at infinity of the system of differential equations conjugate to system (1):

$$\psi_z - (Q'\psi)_z + A'\psi + \bar{B}'\bar{\psi} = 0.$$

**Theorem 2.** *Every solution of equation (1), bounded in the domain  $T$ , is representable in the form*

$$w(z) = \Phi(z) + \iint_T \Gamma_1(z, t) \overline{\Phi(t)} dT + \iint_T \Gamma_2(z, t) \overline{\Phi(t)} dT + \sum_{k=1}^N c_k w_k(z), \quad (6)$$

where  $\Phi(z)$  is a  $Q$ -holomorphic vector;  $\Gamma_1(z, t)$ ,  $\Gamma_2(z, t)$  are matrices depending only on the domain  $T$  and on the coefficients of equation (1);  $c_k$  are real constants ( $c_k = [w, \bar{w}_k]$ ). The vector  $\Phi(z)$  of representation (6) satisfies the conditions

$$\operatorname{Im} \int_L (\Phi, (dt + Q'\bar{d}t)\psi_k) = 0, \quad k = 1, \dots, N. \quad (7)$$

Conversely, if  $\Phi(z)$  is a bounded  $Q$ -holomorphic vector satisfying conditions (7), then formula (6) gives a solution of equation (1).

In addition to the kernels  $\Gamma_1(z, t)$  and  $\Gamma_2(z, t)$ , we also consider the kernels

$$\Omega_1(z, t) = V(t, z) + \iint_K \Gamma_1(z, \sigma) V(t, \sigma) dK_\sigma,$$

$$\Omega_2(z, t) = \iint_K \Gamma_2(z, \sigma) \overline{V(t, \sigma)} dK_\sigma.$$

The kernels  $\Omega_1(z, t)$  and  $\Omega_2(z, t)$ , as well as  $\Gamma_1(z, t)$  and  $\Gamma_2(z, t)$ , are naturally continued to the whole plane in both variables  $z$  and  $t$ . They vanish at infinity. For the kernels  $\Omega_1, \Omega_2, \Gamma_1$ , and  $\Gamma_2$  one derives a number of integral and differential relations analogous to relations (5.17), (5.18), (5.19), (5.24), and (5.25) of the work (2). With the aid of the kernels  $\Omega_1$  and  $\Omega_2$ , Cauchy's formula for system (1) is written down.

**Theorem 3.** In order that the vector  $w(t)$ , prescribed on  $L$  and Hölder-continuous on  $L$ , be the boundary value of a regular solution of system (1) in the domain  $T$ , Hölder-continuous in  $T + L$ , it is necessary and sufficient that:

$$1) \quad \operatorname{Im} \int_L ((dt + Q\bar{d}t)w(t), \psi_k(t)) = 0, \quad k = 1, 2, \dots, N.$$

$$2) \quad \frac{1}{2\pi i} \int_L \left[ \Omega_1(z, t)(dt + Q\bar{d}t)w(t) - \Omega_2(z, t)\overline{(dt + Q\bar{d}t)w(t)} \right] + \sum_{k=1}^N c_k w_k(z) = 0$$

for any point  $z \in \bar{T} + L$  and for some real constants  $c_1, c_2, \dots, c_N$ .

If the complement of the domain  $T = T^+$  is denoted by  $T^-$ , then with the aid of the kernels  $\Omega_1(z, t)$  and  $\Omega_2(z, t)$  one can consider a Cauchy-type integral for system (1), write the usual Plemelj formulas and, in particular, give a solution of the following problem:

Find a piecewise-regular solution of system (1) satisfying the conditions

$$w^+(t) - w^-(t) = \mu(t) \quad \text{on } L, \quad w(\infty) = 0. \quad (8)$$

**Theorem 4.** For the solvability of problem (8) it is necessary and sufficient that

$$\operatorname{Im} \int_L (\psi_k, (dt + Q\bar{d}t)\mu(t)) = 0, \quad k = 1, 2, \dots, N. \quad (9)$$

Using the representations given above, one can transfer to solutions of system (1) a number of other propositions from the theory of I. N. Vekua. These results find application in the theory of boundary-value problems for system (1).

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*Note: Figure translations are in progress. See original paper for figures.*

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