V. I. KOGAN

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Abstract

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V. I. KOGAN

ON THE THEORY OF THE BROADENING OF SPECTRAL LINES IN A PLASMA

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1. The article presents the results of a theory of the broadening of spectral lines due to the Stark effect, taking into account the perturbation of the radiating atom by the electric field of a collection of a large number of moving charged particles. Only in such a scheme can one rigorously consider the relation between the statistical and impact theories, which up to now has been investigated (for the Stark effect) only under the assumption of successive collisions of the atom with individual perturbing particles ([1–4]). This "binary" model is the less adequate to the phenomenon, the larger the radius of the interaction forces of the atom and the particles–something that was taken into account neither in the impact theory nor in its generalizations. Meanwhile, for the linear Stark effect (the longest-range of the broadening mechanisms), the impact theory gives divergent values of the width and, especially, of the shift of the Stark component of a line ([1,2]), which indicates the inapplicability of the binary scheme to this case*.

2. In the proposed theory it is assumed that: a) not only ions but also electrons may be described classically; b) their distribution in space is random at a given mean density; c) the perturbing field is homogeneous; d) its action on the atom is adiabatic. Under these assumptions, a Fourier expansion is made of the wave train emitted by the perturbed equivalent oscillator, which gives the shape of any Stark component in the classical-adiabatic approximation.

Let us introduce the notation: ω_0–the unperturbed frequency, ω–the observed frequency; $x(t)$–the instantaneous frequency shift of the equivalent oscillator; $\mathbf{E}(t)$–the perturbing field; T–the plasma temperature; N–the density of ions (and electrons); $f_i(\mathbf{v})$ and $f_e(\mathbf{v})$–Maxwellian distributions of their velocities; $v_{0i} = (2kT/m_i)^{1/2}$, $v_{0e} = (2kT/m_e)^{1/2}$; α and β–constants of the linear and quadratic Stark effects, defined by the equalities $x_{\text{lin}} = \alpha e^{-1}|\mathbf{E}|$, $x_{\text{kv}} = \beta e^{-2} E^2$. In all the limiting transitions it is understood that $N = \text{const}$.

3. The Fourier expansion mentioned gives the following intensity distribution of a certain Stark component:

$$I(\omega) = \pi^{-1}\,\mathrm{Re}\int_0^\infty \exp[-i(\omega-\omega_0)\tau]\,\varphi(\tau)\,d\tau, \tag{1}$$

where, for the linear Stark effect,

$$\varphi(\tau) = \lim_{M\to\infty} \int \cdots \int \prod_{a=1}^{M} \left(\frac{1}{8\pi^3}\, d\vec{\sigma}_a\, d\vec{\chi}_a\, e^{i\vec{\sigma}_a + i\vec{\chi}_a \vec{\sigma}_a} \right) \exp[-NC(\vec{\chi}_1, \dots, \vec{\chi}_M; \tau)], \tag{2}$$

$$C(\dots) = \iiint \left\{ 1 - \cos\left[\frac{\alpha\tau}{M} \sum_{a=1}^{M} \vec{\chi}_a \frac{(\mathbf{r}_0 + a\mathbf{v}\tau/M)}{|\mathbf{r}_0 + a\mathbf{v}\tau/M|^3} \right] \right\} [f_i(\mathbf{v}) + f_e(\mathbf{v})]\, d\mathbf{v}\, d\mathbf{r}_0. \tag{3}$$

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* Ignoring the divergent shift on the basis of the symmetry of the line as a whole ([1]) is inadmissible, since it is precisely the shifts of the components that determine its effective width.

(integration is over the entire space of the corresponding variables). For the quadratic Stark effect, φ differs from (2) only by replacing σ_a by σ_a^2, and C from (3) by replacing $\alpha\tau/M$ by $(\beta\tau/M)^{1/2}$.

4. Taking into account only ions, we obtain from (1)–(3), for sufficiently large values of the dimensionless parameter $h_i \equiv N(\alpha/v_{0i})^3$ or $(\omega - \omega_0)$:

$$I_{h_i}(\omega) = (\Delta\omega_0)^{-1} \{\mathcal{H}[(\omega - \omega_0)/\Delta\omega_0] + h_i^{-2/3} S[(\omega - \omega_0)/\Delta\omega_0]\}, \tag{4}$$

where $\Delta\omega_0 \equiv 2.60\alpha N^{2/3}$, $\mathcal{H}$ is the Holtsmark function ([1,5]),

$$S(x) = (128\pi^2)^{-1} \left({}^{15}/_{4}\right)^{5/3} x^{-5} \int_0^\infty e^{-(y/x)^{3/2}} [(192 - 42y^2)\cos y +$$

$$+ (111y - 12y^3 - 192y^{-1} - 5x^{3/2} y^{-1/2}) \sin y]\, dy. \tag{5}$$

The function S is characterized by the following table:

x	0.0	0.1	0.2	0.4	0.6	0.8	1.0	1.2	1.4	1.6
$S(x)\cdot 10^2$	−3.66	−3.45	−3.29	−2.66	−1.67	−0.589	+0.376	1.09	1.50	1.64
x	2.0	2.4	2.8	3.2	3.4	3.8	4.2	4.6	5.0	$\to \infty$
$S(x)\cdot 10^4$	134	80.8	37.9	13.0	5.86	−1.60	−3.95	−4.23	−3.71	$-447x^{-1/2}$

The "generalized" statistical approximation (4) supplements the usual one with a correction for the thermal motion of ions. For $h_i \gg 1$ the Holtsmark theory is valid for $\omega - \omega_0 \gg N^{1/3} v_{0i}$, i.e. almost over the entire contour of the component (the calculation of $I(\omega)$ in the region $\omega - \omega_0 \lesssim N^{1/3} v_{0i}$ has not been completed), and for $h_i \ll 1$—on the far wing, at $\omega - \omega_0 \gg v_{0i}^2 \alpha^{-1}$, where (4) takes the form

$$I(\omega) = 2\pi\alpha^{3/2} N(\omega - \omega_0)^{-5/2} \left[1 - {}^5\!/_{64} v_{0i}^2 \alpha^{-1} (\omega - \omega_0)^{-1}\right]. \tag{6}$$

Formula (6) coincides with the result of the binary calculation ([6]) for the line wing and a perturbation of the form $\chi(t) \sim [r(t)]^{-n}$, if in that result one sets $n = 2$ and averages it over velocities.

5. For small values of h_i (the "quasibinary" case) we obtain

$$I_{h_i}(\omega) = \alpha v_{0i}^{-2} F_{h_i}[(\omega - \omega_0 - \lambda_1 \Delta\omega_0)\alpha v_{0i}^{-2}], \tag{7}$$

where $\lambda_1 = 3.41$ is the first moment of the Holtsmark distribution,

$$F_{h_i}(x) = \pi^{-1} \operatorname{Re} \int_0^\infty \exp\left[-ixy + h_{iB}\left(\frac{1}{y}\right)\right] dy, \tag{8}$$

$$B(t) = 4\pi^{1/2} \Big[0.444t^{-3} - 0.96it^{-2} - 1.33t^{-3/2} e^{i\pi/4} \int_0^z K_0(x)\, dx + 2.67it^{-2} K_0(z) -$$

$$-5.92it^{-2} K_0(1.83z) - 1.33t^{-3/2} e^{i\pi/4} (1 - 2it^{-1}) K_1(z) - 3.24t^{-5/2} e^{3\pi i/4} K_1(1.83z)\Big] \tag{9}$$

(K_0, K_1 are Macdonald functions; $z \equiv 2\sqrt{t}\, e^{-i\pi/4}$). For lack of space we do not give tables of the functions $F_{h_i}(x)$. For $(\omega - \omega_0 - \lambda_1 \Delta\omega_0) \gg v_{0i}^2 \alpha^{-1}$, (7) takes the form (6); consequently, the quasibinary and generalized statistical approximations are not entirely alternatives. For $|\omega - \omega_0 - \lambda_1 \Delta\omega_0| \ll v_{0i}^2 \alpha^{-1}$ (the predominant, "quasi-impact" part of the component) we have

$$F_{h_i}(x) \approx \pi^{-1} \int_0^\infty \cos[(x + \pi^{1/2} h_i) y] \exp[-2\pi^{1/2} h_i y \ln(0.520y)]\, dy \tag{10}$$

and the corresponding $I_{h_i}(\omega)$ according to (7). The shift of the maximum of the curve (7), (10) is equal to $\omega - \omega_0 \simeq \lambda_1 \Delta\omega_0$, while its width is $\sim N\alpha^2 v_{0i}^{-1} \ll \Delta\omega_0$. For $v_{0i} \to \infty$ we have $I_{h_i}(\omega) \to \delta(\omega - \omega_0 - \lambda_1 \Delta\omega_0)$, i.e., a static shift of the component corresponding to the field $\overline{E} = \lambda_1 \cdot 2.60 e N^{2/3}$ averaged over the Holtsmark distribution.

Thus, for $h_i \ll 1$ the action of the ensemble of moving ions reduces (within the framework of the adiabatic model), in the main, to a "static"Stark splitting of the line by the mean field, accompanied by impact broadening of the components. The principal effect—the splitting—can also be obtained directly: exactly, if the rule for calculating the shift given in ([1],[2]) is generalized, and, as to order of magnitude, if instead of arbitrarily discarding the divergent shift of a component in the impact theory ([1],[2]), the corresponding integral is cut off at distances $\sim N^{-1/3}$ (allowing for the mutual compensation of the influence of more distant fly-bys). The principal qualitative result consists in the fact that the effective width of the line as a whole for $h_i \ll 1$ is determined by the same dependence as for $h_i \gg 1$: $\delta\omega \sim \bar{\alpha} N^{2/3}$ ($\bar{\alpha}$ is a certain effective Stark constant of the line), contrary to the existing (impact) theory, according to which $\delta\omega \sim N(\bar{\alpha})^2 v_{0i}^{-1}$.

6. The result of item 5 points to the unfoundedness of the usual assertion that the broadening influence of electrons is small in comparison with that of ions in the linear Stark effect. We shall therefore include in (3) also the "electron" term. For $h_i \gg h_e \gg 1$ or $h_e \ll h_i \ll 1$ $\left(h_e = N(\alpha/v_{0e})^3\right)$, $I(\omega)$ is obtained by replacing N by $2N$ in the results of items 4 and 5. The most interesting case is $h_i \gg 1,\ h_e \ll 1$. In view of the crudeness of our scheme as applied to electrons, we shall restrict ourselves to the limiting case $v_{0i} = 0,\ v_{0e} = \infty$, which gives

$$\begin{aligned} &I(\omega) \equiv 0 && \text{for } \omega - \omega_0 \leqslant \lambda_1 \Delta\omega_0, \\ &I(\omega) = (\Delta\omega_0)^{-1} L\big[(\omega-\omega_0)/\Delta\omega_0\big] && \text{for } \omega - \omega_0 \geqslant \lambda_1 \Delta\omega_0, \end{aligned} \tag{11}$$

where the function L is expressed in a complicated way through $\mathcal{H}$; tabulation gives:

x	3.41	3.42	3.46	3.52	3.58	3.64	3.76	4.0	4.4	5.0	6.0	8.0	10	20	50
$L(x)\cdot 10^3$	0	245	510	673	745	780	754	616	380	203	80	19	8.5	1.9	0.1

The curve (11) is the crude result of the superposition of two effects—the "electron" shift and the "ion" broadening. For $\omega - \omega_0 \gg \lambda_1 \Delta\omega_0$, (11) reduces to the leading term (6), while for $\omega - \omega_0 \lesssim \lambda_1 \Delta\omega_0$ the difference between (11) and the Holtsmark formula is very large. Of course, the broad dip in the inner region should in reality be filled in to a significant extent—above all, owing to the strong nonadiabaticity of the action of electrons on the atom, which leads to a "mixing" of the Stark components. On the whole, the result of item 6 is a qualitative indication that the role of electrons in the broadening of hydrogen lines is comparable with the role of ions.

7. For the quadratic Stark effect it is sufficient to take into account perturbing particles of one kind. For $v_0 \to 0$ we obtain the corresponding Holtsmark formula. In the most interesting "binary" case $g \equiv N\beta/v_0 \ll 1$

$$\Phi(\tau) \simeq \exp\left\{-2\pi N \int f(\mathbf{v})\, d\mathbf{v} \int_0^\infty \rho\, d\rho \int_{-\infty}^\infty dz \left[1 - \exp\left(i\beta \int_0^\tau [\rho^2 + (z+vt)^2]^{-2}\, dt\right)\right]\right\}.$$

This result coincides with that obtained by formal application to the case under consideration ($\chi_j = \beta r_j^{-4}$) of the Lenz–Anderson theory ([7],[3]), based on a "scalar" addition of perturbations from individual particles: $\chi = \chi_1 + \chi_2 + ...$ This is natural, since for $g \ll 1$ the average number of perturbing particles inside the Weisskopf radius is small, so that the character of the addition of perturbations plays no role. Thus, the "scalar" theory ([7],[3]) (strictly valid only for the case $\chi_j \sim r_j^{-6}$ and not applicable to the case $\chi_j \sim r_j^{-2}$) is, for $\chi_j \sim r_j^{-4}$, equivalent to the binary approximation of the "vector" theory set forth,

i.e., based on the superposition of fields, theory. For $\omega - \omega_0 \gg v_0^{4/3}\beta^{-1/3}$ (the wing), a statistical formula is obtained for $I(\omega)$, while for $\omega - \omega_0 \ll v_0^{4/3}\beta^{-1/3}$ and further "to the left" (the predominant part of the line), an impact formula is obtained, coinciding with the usual one when it is taken into account that $\overline{v}^{1/3} = 1.02 v_0^{1/3}$. For the form of the "negative" (low-intensity) wing see ([3]).

Analysis shows that even for $g \ll 1$ the binary approximation a, and hence also the impact theory, are, strictly speaking, invalid near the maximum of $I(\omega)$: the simultaneous action of many particles plays a role here.

8. Let us discuss the validity of the assumptions of Sec. 2 (all estimates refer to hydrogen; n is the principal quantum number of the upper level; everywhere N is in cm^{-3}, T in eV). Assumption a), apparently, is valid for $T \gg 13.5n^{-4}$ (see ([8])); b) is valid, as follows from ([9]), for $T \gg 2\cdot 10^{-5} N^{1/3}$; c) is essential only for the quadratic Stark effect ([2]). For the linear effect, however, assumption d) is a serious limitation of the theory set forth. In the "statistical" region the adiabatic approximation is legitimate ([1,2]); therefore the results of Sec. 4 (and Sec. 5 for the wing) are quantitatively reliable. In the "quasi-impact" region, however, the adiabaticity condition is violated. True, for ions the effect of nonadiabaticity is to a considerable extent compensated by the effect of rotation of the radiating atom ([10]); therefore the corresponding results of Sec. 5 are qualitatively valid. But for electrons the effective flight times ρ/v_e are not only small compared with $(\omega - \omega_0)^{-1}$ (the usual criterion of nonadiabaticity), but are also comparable with the period of revolution of the electron in the excited atom ([11]). Taking $\rho_{\text{eff}} \sim (e/\bar{E})^{1/2}$, we find that these times become comparable when $T^{1/2}N^{1/2}n^3 \sim 3\cdot 10^7$. Taking this effect into account clearly lies outside the scope of our theory, so that the result of Sec. 6 has no more than qualitative significance. It should, however, be emphasized that in Sec. 6 the simultaneous action of many "distant" electrons (an effect $\sim N^{2/3}$) has been taken into account, whereas existing (including quantum) theories of impact broadening by electrons (see ([4])) are of a purely binary character, i.e., take into account the influence only of the nearest electrons (an

effect $\sim N/v_{0e}$)* . What has been said gives reason to believe that these theories do not describe the main ($\sim N^{2/3}$) effect of electron broadening.

From Secs. 4 and 5 it follows that for $h_i > 0.004$ the influence of ions is, in the main, statistical, while for $h_i < 0.004$ it is quasi-impact. The equality $h_i = 0.004$ corresponds to the following: $T \approx 1 \cdot 10^{-11}(\bar{\alpha})^2 N^{2/3}$; for the values of $\bar{\alpha}$ for the Balmer lines see ([1]). The line width in the Holtsmark case is $\delta\omega$ (sec^{-1}) $\approx 15.6\bar{\alpha}N^{2/3}$; in the quasi-impact case $\delta\omega \approx 17.7\bar{\alpha}N^{2/3}$; with electrons taken into account (according to (11)) $\delta\omega \approx 23\bar{\alpha}N^{2/3}$.

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Moscow Engineering-Physics Institute

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* Theory ([12]) is also, in essence, equivalent to the binary approximation.

Note: Figure translations are in progress. See original paper for figures.