ON THE QUESTION OF THE INVERSE PROBLEM OF POTENTIAL THEORY

1958

SovietRxiv

Abstract

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MATHEMATICS

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ON THE QUESTION OF THE INVERSE PROBLEM OF POTENTIAL THEORY

(Presented by Academician S. L. Sobolev on June 27, 1957)

In the dissertation of V. K. Ivanov ([1]), an investigation was begun of the question of the change in a region producing a given exterior potential when the density of the matter filling it is increased. In the present note a qualitative characterization of this change is given; namely, it is proved that under a monotone increase of the density the region is monotonically compressed. The method used by us was proposed by V. P. Simonov ([3]) for proving uniqueness theorems.

Theorem 1. *Let the regions D_1 and D_2, star-shaped with respect to the pole O, be filled with matter of constant densities μ_1 and μ_2, respectively, where $\mu_1 > \mu_2 > 0$. If, moreover, the exterior potentials of the regions are identically equal, then*

$$D_1 \subset D_2.$$

Proof. Let the boundaries of the regions D_1 and D_2 have, in polar coordinates (r, φ), the equations $r = r_1(\varphi)$ and $r = r_2(\varphi)$, respectively, and let

$$R_1(\varphi) = \min\{r_1(\varphi), r_2(\varphi)\}; \qquad R_2(\varphi) = \max\{r_1(\varphi), r_2(\varphi)\}.$$

Suppose that the inclusion $D_1 \subset D_2$ is not satisfied (the impossibility of the reverse inclusion is obvious). This means that on some set $A \subset [0, 2\pi]$ we have $r_2(\varphi) < r_1(\varphi)$. For the proof it is sufficient ([2,3]) to construct in the region $D_1 \cup D_2$ such a harmonic function U that

$$\iint_{D_1} U\mu_1\, d\sigma - \iint_{D_2} U\mu_2\, d\sigma > 0. \tag{1}$$

Consider in the region $D_1 \cup D_2$ a bounded harmonic function $V(r, \varphi)$ taking on the boundary the values

$$V(R_2, \varphi) = \begin{cases} 1, & \varphi \in A, \\ 0, & \varphi \in B = [0, 2\pi] \setminus A. \end{cases}$$

We show that the function

$$U = 2V + r\frac{\partial V}{\partial r},$$

which is harmonic together with V, will satisfy inequality (1). The left-hand side of this inequality is equal to

$$\mu_1 \iint_{D_1 \setminus D_2} U\, d\sigma - \mu_2 \iint_{D_2 \setminus D_1} U\, d\sigma + (\mu_1 - \mu_2) \iint_{D_1 \cup D_2} U\, d\sigma =$$

$$= \mu_1 I_1 + \mu_2 I_2 + (\mu_1 - \mu_2) I_3.$$

It is easy to verify that each of the integrals I_1, I_2, I_3 is positive. Indeed,

$$I_1 = \iint_{D_1 \setminus D_2} U\, d\sigma = \int_A \int_{R_1}^{R_2} U r\, dr\, d\varphi = \int_A [R_2^2 - R_1^2 V(R_1, \varphi)]\, d\varphi > 0,$$

$$I_2 = -\iint_{D_2 \setminus D_1} U\, d\sigma = \int_B R_1^2 V(R_1, \varphi)\, d\varphi > 0,$$

$$I_3 = \iint_{D_1 \cup D_2} U\, d\sigma = \int_0^{2\pi} R_1^2 V(R_1, \varphi)\, d\varphi > 0.$$

The theorem is proved.

Remark. The formulation and proof of Theorem 1 carry over without changes to the three-dimensional case.

Theorem 2. *Let D_1 be a plane simply connected domain bounded by an analytic curve and producing, when filled with masses of density $\mu_1(x, y) > 0$, the exterior potential V. Then there exists a domain G, containing D_1, such that for any simply connected domain $D_2 \subset G$, having, with positive density $\mu_2(x, y) < \mu_1(x, y)$, the same exterior potential V, we always have*

$$D_1 \subset D_2.$$

For the proof we again assume the contrary. Let G be the largest domain to which the function $w = f(z)$, mapping D_1 conformally onto the disk $|w| < 1$, can be continued; let $U(x, y)$ be a harmonic function, defined and positive in the domain D_1 and equal to zero on those parts of its boundary that lie in the closure of D_2. From the conditions of the theorem it follows that $U(x, y)$ continues to the domain D_2, and all its values in $D_2 \setminus D_1$ are negative. Therefore

$$\iint_{D_1} U\mu_1 \, d\sigma - \iint_{D_2} U\mu_2 \, d\sigma > 0,$$

which proves the theorem.

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Received
25 VI 1957

REFERENCES

1. V. K. Ivanov, Doctoral dissertation, Math. Inst. named after V. A. Steklov, Academy of Sciences of the USSR, 1955. P. S. Novikov, DAN, 18, No. 3, 165 (1938). V. P. Simonov, Candidate dissertation, Moscow State Pedagogical Institute named after V. I. Lenin, 1954.

Note: Figure translations are in progress. See original paper for figures.