MATHEMATICS

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

MATHEMATICS

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ON THE EXPEDIENCY OF USING THE ALTERNATING SCHWARZ METHOD ON ELECTRONIC DIGITAL COMPUTERS

(Presented by Academician S. L. Sobolev, 13 XII 1957)

1. Consider the difference analogue of the Dirichlet problem for the Laplace equation in the case of a rectangular domain (Fig. 1)

$$\Delta^h U = 0 \quad \text{in } G; \qquad U\big|_{\Gamma} = \psi. \tag{1}$$

Suppose that the number of grid points $M = mn$ of the domain satisfies the conditions:

$$N_2 - N_1 < mn, \quad N_2 - N_1 \geq m\left(\frac{n}{2} + 1\right) \quad \text{for even } n; \tag{1'}$$

$$N_2 - N_1 < mn, \quad N_2 - N_1 \geq m\left(\frac{n}{2} + \frac{3}{2}\right) \quad \text{for odd } n, \tag{1''}$$

where N_1 is the number of commands in the immediate-computation subprogram, and N_2 is the capacity of the internal storage device. The set of grid points M of such a domain can be divided into two equal connected subsets M_1 and M_2, which satisfy the conditions

$$M_1 = M_2 \leq N_2 - N_1, \qquad M_1 \cup M_2 = M,$$

$$M_1 \cap M_2 \geq 2m.$$

Fig. 1

To solve the problem one may use two methods: a) carry out each Liebmann iteration successively in both subsets; b) apply the alternating Schwarz method. To determine the expediency of using the latter method, we shall establish the rate of convergence of the Schwarz iterative process.

2. Let $U^{(k)}(M)$ be the value of U at the point M after the k-th Schwarz iteration. Consider the function $\omega^{(k)}(M) = U^{(k)}(M) - U^{(k-1)}(M)$. It is the solution of the following problem (for the domain $ABMN$):

$$\Delta^h \omega^{(k)} = 0 \quad \text{in } ABMN;$$

$$\omega^{(k)}\big|_{AB} = \omega^{(k)}\big|_{BM} = \omega^{(k)}\big|_{AN} = 0; \qquad \omega^{(k)}\big|_{MN} = d^{(k-1)}(M), \tag{2}$$

where $d^{(k-1)}(M) = \omega^{(k-1)}\big|_{MN}$ is the residual on the column MN in the preceding Schwarz iteration in the domain $LCDE$.

Denote by $\bar{d}^{(k)}$ the maximum of the modulus of $d^{(k)}(M)$. Consider the following problem

$$\Delta^h \bar{\omega}^{(k)} = 0 \quad \text{in } ABMN;$$

$$\bar{\omega}^{(k)}\big|_{AB} = 0; \quad \bar{\omega}^{(k)}\big|_{MN} = \bar{d}^{(k-1)}; \qquad \bar{\omega}^{(k)}\big|_{AN} = \bar{\omega}^{(k)}\big|_{BM} = \frac{j}{n_1}\bar{d}^{(k-1)}, \tag{3}$$

where j is the column number, counting AB as zero; $\bar{\omega}^{(k)}$ is a majo-

boundary for $\omega^{(k)}$. The solution of the boundary-value problem (3) will be $\bar{\omega}^{(k)} = \dfrac{i}{n_1}\bar{d}^{(k-1)}$. Since $\bar{\omega}^{(k)} \geqslant \omega^{(k)}$, for the column LE we have

$$\omega^{(k)}(M)\big|_{LE} \leqslant \frac{n_2}{n_1}\bar{d}^{(k-1)}. \tag{4}$$

Considering the function $\omega^{(k)}(M)$ in the domain $LCDE$ and using inequality (4), we find

$$\omega^{(k)}(M)\big|_{MN} \leqslant \left(\frac{n_2}{n_1}\right)^2 \bar{d}^{k-1}. \tag{5}$$

The next iteration in $ABMN$ gives, for the column LE,

$$\omega^{(k+1)}(M)\big|_{LE} \leqslant \frac{n_2}{n_1}\bar{d}^{(k)}, \tag{6}$$

where

$$\bar{d}^{(k)} = \max[d^{(k)}(M)\big|_{MN}] = \max[\omega^{(k)}(M)\big|_{MN}] \leqslant \left(\frac{n_2}{n_1}\right)^2 \bar{d}^{(k-1)}. \tag{7}$$

Substituting (7) into (6), we obtain

$$\omega^{(k+1)}(M)\big|_{LE} \leqslant \left(\frac{n_2}{n_1}\right)^3 \bar{d}^{\,k-1}. \tag{8}$$

Without loss of generality one may assume

$$\max[\omega^0(M)\big|_{MN}] = \max[\{U^0(M) - U_1(M)\}\big|_{MN}] \leqslant \max_G[U^0(M) - U_1(M)] \leqslant 1, \tag{9}$$

where $U_1(M)$ is the first approximation, and then we obtain

$$\omega^{(k)} \leqslant \left[\left(\frac{n_2}{n_1}\right)^2\right]^{k-1}. \tag{10}$$

$(n_2/n_1)^2$ will play the role of the spectral norm of the eigenvalues in Schwarz' s iterative process.

Estimate (10) is valid for any domain contained in the given rectangle. It is attained if the smallest height of a column of the domain, not counting the end ones, is approximately equal to n. In the case where the domain has narrow necks, estimate (10) may give rather overstated values.

3. The number of Schwarz iterations S necessary for solving problem (1) with accuracy 10^{-k}, under condition (9), is equal to the least integer

$$S \geqslant \left|\frac{R}{2\ln(n_2/n_1)}\right|, \qquad \text{where } R = \ln\left(\frac{1.6 \cdot 10^{-k+1}}{m^2 + n^2}\right).$$

The number of Liebmann iterations $H_{(0)}$ in each part of the domain under condition (9) will be (3)

$$H_{(0)} \geqslant \left|\frac{R}{\ln K^*}\right| \simeq \frac{R}{\frac{1}{2}\pi^2(m^{-2} + n_1^{-2})}.$$

For one Schwarz iteration it is necessary to carry out $2H_{(0)}$ Liebmann iterations and to access the buffer storage device 4 times. The total time for solving problem (1) by applying the Schwarz method will be

$$T_1 \leqslant 2SH_{(0)}(m-1)(n_1-1)L\tau_1 + 4S\tau_2, \tag{11}$$

where L is the number of arithmetic operations in the operator Δ^h; τ_1 is the time for performing one arithmetic operation; τ_2 is the time for reading (mn_1) numbers from the storage device.

Estimate (11) is too high. It can be improved by taking into account the fact that the number of Liebmann iterations required to achieve the required accuracy becomes smaller with each Schwarz iteration. The first Schwarz iteration under condition (9) requires, for one part of the domain, $H_{(0)}$ Liebmann iterations. In the second Schwarz iteration one may assume that $\left|{}_2U^0 - U\right| \leqslant (n_2/n_1)^2$, where ${}_2U^0$ is the initial (mn)-dimensional

the vector for the second Schwarz iteration. The number of Liebmann iterations $H_{(1)}$ and the total time $T^{(1)}$ of the second Schwarz iteration will be equal to

$$H_{(1)} = \frac{\left|R + \ln(n_1/n_2)^2\right|}{\frac{1}{2}\pi^2(m^{-2}+n_1^{-2})}, \qquad T^{(1)} = 2H_{(1)}(m-1)(n_1-1)L\tau_1 + 4\tau_2,$$

and at the k-th Schwarz iteration

$$H_{(k-1)} = \frac{\left|R + \ln(n_1/n_2)^{2(k-1)}\right|}{\frac{1}{2}\pi^2(m^{-2}+n_1^{-2})} = \frac{|R| - \left|\ln(n_1/n_2)^{2(k-1)}\right|}{\frac{1}{2}\pi^2(m^{-2}+n_1^{-2})},$$

$$T^{(k-1)} = 2H_{(k-1)}(m-1)(n_1-1)L\tau_1 + 4\tau_2.$$

The passage in the expression for $H_{(k-1)}$ from the absolute value of the sum to the difference of the absolute values is justified, since the following inequalities hold:

$$\operatorname{sign} R \neq \operatorname{sign} \ln(n_1/n_2); \quad |R| > |2(k-1)\ln(n_1/n_2)|.$$

The time of the first S Schwarz iterations will be

$$T_2 = \sum_{k=0}^{S-1} T^{(k)} = 2\sum_{k=0}^{S-1} H_{(k)}(m-1)(n_1-1)L\tau_1 + 4S\tau_2 =$$

$$= \left[\frac{2S|R|}{\frac{1}{2}\pi^2(m^{-2}+n_1^{-2})} - \frac{\left|2\sum_{k=0}^{S-1}\ln(n_1/n_2)^{2k}\right|}{\frac{1}{2}\pi^2(m^{-2}+n_1^{-2})}\right](m-1)(n_1-1)L\tau_1 + 4S\tau_2 =$$

$$= H_0(S+1)(m-1)(n_1-1)L\tau_1 + 4S\tau_2. \tag{12}$$

4. Let us consider the following combined method for solving the boundary-value problem. At the k-th Schwarz iteration, the Liebmann iteration process is continued until the residuals become less than $(n_2/n_1)^{2k}$. After S iterations the residuals will be, in modulus, less than $\exp R$, which for the given rectangular region corresponds to solving problem (1) with accuracy 10^{-k}. Then the number of Liebmann iterations in one part of the region at the first Schwarz iteration will be

$$\overline{H}_{(0)} = \left| \frac{2\ln(n_2/n_1)}{\frac{1}{2}\pi^2(m^{-2}+n_1^{-2})} \right|,$$

and at the k-th iteration

$$\overline{H}_{(k)} = \left| \frac{\ln\left[(n_2/n_1)^{2k}(n_1/n_2)^{2(k-1)}\right]}{\frac{1}{2}\pi^2(m^{-2}+n_1^{-2})} \right| = \overline{H}_0.$$

The total computation time by the above **combined** method is equal to

$$T_3 = 2S\overline{H}_{(0)}(m-1)(n_1-1)L\tau_1 + 4S\tau_2 =$$

$$= \frac{2|R|}{\frac{1}{2}\pi^2(m^{-2}+n_1^{-2})}(m-1)(n_1-1)L\tau_1 + 4S\tau_2 =$$

$$= 2H_0(m-1)(n_1-1)L\tau_1 + 4S\tau_2. \tag{13}$$

5. The total time for solving the boundary-value problem without applying the Schwarz method is equal to

$$T = H[(m-1)(n-1)L\tau_1 + 4\tau_3], \tag{14}$$

where τ_3 is the time for reading from the storage device $m(n+2)/2$ numbers, and H is the number of Liebmann iterations for the whole region,

$$H = \frac{|R|}{\frac{1}{2}\pi^2(m^{-2}+n^{-2})}.$$

6. In the case where, instead of Liebmann iteration, the overrelaxation method is used, T, T_1, T_2, and T_3 take the following values:

$$T = |R|\frac{1}{\pi\sqrt{2}\sqrt{m^{-2}+n^{-2}}}[(m-1)(n-1)(L+3)\tau_1 + 4\tau_3],$$

$$T_1 = \sqrt{2}\,S|R|\frac{(m-1)(n_1-1)(L+3)\tau_1}{\pi\sqrt{m^{-2}+n_1^{-2}}} + 4S\tau_2,$$

$$T_2 = (S+1)|R|\frac{(m-1)(n_1-1)(L+3)\tau_1}{\pi\sqrt{2}\sqrt{m^{-2}+n_1^{-2}}} + 4S\tau_2,$$

$$T_3 = \sqrt{2}|R|\frac{(m-1)(n_1-1)(L+3)\tau_1}{\pi\sqrt{m^{-2}+n_1^{-2}}} + 4S\tau_2.$$

Table 1

	$m =$ 20	$m =$ 20	$m =$ 20	$m =$ 20	$m =$ 25	$m =$ 25	$m =$ 25	$m =$ 30	$m =$ 30	$m =$ 30	$m =$ 30
	$n =$ 50	60	70	80	50	60	70	40	45	50	55
$T \cdot 10^{-3}$ (Liebmann)	6,4	8,1	9,9	11,6	11,7	15,1	18,6	13,1	15,8	18,9	22,3
$T \cdot 10^{-2}$ (over-relax.)	3,9	4,8	5,8	6,8	5,9	7,4	8,9	6,3	7,1	8,2	9,3
$T_1 \cdot 10^{-4}$ (Liebmann)	2,5	4,6	8,9	21,3	6,2	15,0	108,8	6,2	9,7	16,9	38,6
$T_1 \cdot 10^{-3}$ (over-relax.)	1,6	2,9	5,6	13,3	3,6	8,8	63,8	3,3	5,2	9,1	20,5
$T_2 \cdot 10^{-3}$ (Liebmann)	1,5	2,6	4,7	10,9	3,4	7,8	54,8	3,5	5,2	8,8	19,4
$T_2 \cdot 10^{-3}$ (over-relax.)	1,0	1,6	3,0	7,0	2,0	4,6	32,4	1,9	2,8	4,8	10,6

	$m = 20$	$m = 20$	$m = 20$	$m = 20$	$m = 25$	$m = 25$	$m = 25$	$m = 30$	$m = 30$	$m = 30$	$m = 30$
$T_3 \cdot 10^{-3}$ (Liebmann)	5,1	5,2	5,3	5,5	6,2	6,4	7,1	6,9	7,0	7,2	7,3
$T_3 \cdot 10^{-2}$ (overrelax.)	3,4	3,6	4,0	5,1	4,0	4,7	11,6	4,1	4,3	4,8	6,3

7. Table 1 gives the values of T, T_1, T_2, T_3 in seconds for Liebmann iteration and for the overrelaxation method, computed for the BESM with $k = 6$. A magnetic drum is considered as the buffer storage device. Then

$$\tau_2 = c + mn_1\tau, \qquad \tau_3 = c + \frac{m(n+2)}{2}\tau,$$

where $c = 1/30$ sec. is the time for a half-revolution of the drum; $\tau = 1/800$ sec. is the time for reading one number; $\tau_1 = 10^{-4}$ sec.; $N_2 - N_1 = 902$; L under full automation (2) is equal to 24.

One may dispense with conditions (1′) and (1″) and divide the set M into a larger number of subsets. The iterative process will converge (1), but apparently the rate of convergence will decrease rapidly as the number of constituent subsets is increased.

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Note: Figure translations are in progress. See original paper for figures.