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Abstract

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THE FUNDAMENTAL SOLUTION OF A LINEAR PARTIAL DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

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1. We consider the linear equation with constant coefficients:

$$L\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_m}\right) K(x_1, \dots, x_m) = \delta(x_1, \dots, x_m). \tag{1}$$

If, in the operator L, one replaces $\partial/\partial x_1, \dots, \partial/\partial x_m$ by $\xi_1, \dots, \xi_m$, then one obtains a polynomial of degree n. We require that the polynomial $L(\xi_1, \dots, \xi_m)$ be homogeneous and that the cone $L(\xi_1, \dots, \xi_m) = 0$ have no singular points except the origin of coordinates, i.e., that for $L(\xi_1, \dots, \xi_m) = 0$ and

$$\sum_{i=1}^{m} \xi_i^2 > 0$$

the gradient of L be greater than zero.

Using the solution of equation (1)—the generalized function $K(x_1, \dots, x_m)$—one can obtain a solution of the equation

$$L\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_m}\right) u(x_1, \dots, x_m) = f(x_1, \dots, x_m) \tag{2}$$

by the formula

$$u(\xi_1, \dots, \xi_m) = [K,\ f(x_1 - \xi_1,\ x_2 - \xi_2, \dots,\ x_m - \xi_m)],$$

or, writing the scalar product in integral form:

$$u(\xi_1, \dots, \xi_m) = \int \cdots \int K(x_1, \dots, x_m) f(\xi_1 - x_1, \dots, \xi_m - x_m)\, dx_1 dx_2 \cdots dx_m.$$

Here $f(x_1, \dots, x_m)$ is a finite infinitely differentiable function. The function $K(x_1, \dots, x_m)$ is called the **fundamental solution** of equation (2).

2. It is known (the Kovalevskaya theorem) that $K(x_1, \dots, x_m)$ is an analytic function throughout the whole space, except for the cone formed by those points $x_1, \dots, x_m$ for which the plane

$$\sum_{i=1}^{m} x_i \xi_i = 0$$

is tangent, in ξ-space, to the cone $L(\xi_1, \dots, \xi_m) = 0$. The cone on which $K(x_1, \dots, x_m)$ has singularities is called the **characteristic cone**.

Until now, the behavior of $K(x_1, \dots, x_m)$ in a neighborhood of the characteristic cone had been known only for a few special cases. Only in the work of A. M. Davydova (1) was it proved that, for hyperbolic equations under certain conditions, the fundamental solution itself, or one of its derivatives, tends to infinity when approaching the characteristic cone.

In the present note we give an expansion of the fundamental solution in powers of the distance from the characteristic cone, under the condition that the point of the characteristic cone in whose neighborhood the expansion is given is not singular. The operator L is subject to the conditions formulated at the beginning of the note.

3. Let us indicate a formula for the fundamental solution of equation (2). In the case when the operator $L(\partial/\partial x_1, \dots, \partial/\partial x_m)$ is elliptic, i.e. when

$$L(\xi_1, \dots, \xi_m) \neq 0 \quad \text{for} \quad \sum_{i=1}^{m} \xi_i^2 > 0,$$

the function $K(x_1, \dots, x_m)$ has the form (see (2), p. 36)

$$K(x_1, \dots, x_m) = c_{m,n} \int_{\Omega} \frac{f_{m,n}(\sum x_i \omega_i)\, d\Omega}{L(\omega_1, \dots, \omega_m)}. \tag{3}$$

Here the integration is carried out over the surface of the sphere of unit radius; $d\Omega$ is the surface element of this sphere, $c_{m,n}$ is a constant coefficient depending on the order of the equation and on the dimension of the space, and the function $f_{m,n}(\sum x_i \omega_i)$ has the form:

if $n \geq m$ and m is even, then $f_{m,n}(x) = x^{n-m} \ln |x|$;

if $n \geq m$ and m is odd, then $f_{m,n}(x) = x^{n-m} \operatorname{sign} x$;

if $n < m$ and m is even, then $f_{m,n}(x) = x^{n-m}$;

if $n < m$ and m is odd, then $f_{m,n}(x) = \delta^{(m-n-1)}(x)$.

To prove that the function (3) is indeed a solution of equation (1), it is enough to apply the operator $L(\partial/\partial x_1, \dots, \partial/\partial x_m)$ to the function $K(x_1, \dots, x_m)$ defined by formula (3), and then to use the formula for the expansion of the δ-function into plane waves (see (2), pp. 31–32).

In the case when the operator $L(\partial/\partial x_1, \dots, \partial/\partial x_m)$ is not elliptic, i.e. when there exist $\omega_1, \dots, \omega_m$ such that $L(\omega_1, \dots, \omega_m) = 0$, we cannot apply formula (3) directly, since the integral will diverge. In this case we have to regularize the integral (3). If, as we assumed at the beginning of the note, the cone $L(\omega_1, \dots, \omega_m) = 0$ has no singular points, we can apply Cauchy regularization:

$$K(x_1, \dots, x_m) = \lim_{\varepsilon \to 0} K_\varepsilon(x_1, \dots, x_m), \tag{4}$$

where

$$K_\varepsilon(x_1, \dots, x_m) = c_{m,n} \int_{\Omega_\varepsilon} \frac{F_{m,n}(\sum x_i \omega_i)\, d\Omega}{L(\omega_1, \dots, \omega_m)}.$$

Here Ω_ε is the set of those points on the unit sphere $\sum_{i=1}^m \omega_i^2 = 1$ for which $|L(\omega_1, \dots, \omega_m)| > \varepsilon$. It is easy to prove that the indicated limit actually exists and gives a solution of equation (1).

Formula (4) was used by the author to obtain an expansion of $K(x_1, \dots, x_m)$ in a neighborhood of the characteristic cone.

4. In view of the homogeneity of $K(x_1, \dots, x_m)$, it is enough to give an expansion of $K(x_1, \dots, x_m)$ in a neighborhood of a point of the characteristic cone lying at unit distance from the origin. Let A be such a point; let the line Q be perpendicular to the characteristic cone and pass through A. Denote by s the distance from a certain point of the line Q to the point A. If $K(x_1, \dots, x_m)$ is regarded on the line Q as a function of s, then the following expansion turns out to be valid:

$$K(s) = \Phi(s) + \sum_{j=1}^{\infty} f_j(s).$$

The function $\Phi(s)$ is an analytic function of s in a neighborhood of zero. All singularities of $K(x_1, \dots, x_m)$ in a neighborhood of the point are contained in the series

$$f_1(s) + f_2(s) + \dots + f_i(s) + \dots \tag{5}$$

The general term of this series $f_i(s)$ is always a homogeneous or associated homogeneous generalized function[(2)] of dimension $n - m/2 + j - 2$. If $n > m/2$, then $f_1(s)$, and hence also $f_i(s)$ for $i > 1$, turn out to be ordinary functions, and in this case the fundamental solution $K(x_1, \dots, x_m)$ is also an ordinary function. We note that, directly from formula (3) with the regularization (4), it follows that the fundamental solution is an ordinary function under the more stringent condition $n \gg m$.

5. To indicate the concrete form of the series (5), it is necessary to carry out some preliminary constructions.

Let $x_1, \dots, x_m$ be the point of the characteristic cone in whose neighborhood we seek the expansion. We introduce a coordinate system in such a way that this point has coordinates $y_1 = 1,\ y_2 = y_3 = \cdots = y_m = 0$, and so that the point with coordinates $(1, 0, \dots, 0, y_m)$ coincides with that point on the line Q for which $s = y_m$. It is obvious that the axis y_m will be perpendicular to the characteristic cone at our point and that $K(s) = K(1, 0, \dots, 0, y_m)$.

In a neighborhood of the point $(x_1, \dots, x_m)$ we can write the equation of the characteristic cone in the form

$$y_m = y_m(y_1, \dots, y_{m-1}),$$

where $\partial y_m / \partial y_i$ will be equal to zero at this point for every $i < n$. Form the matrix

$$b_{ij} = \frac{\partial^2 y_m}{\partial y_i \partial y_j} \qquad (i,\ j = 2,\ 3, \dots,\ m-1).$$

Let k be the signature of this matrix (i.e., the number of positive terms c_{ii} in the diagonal matrix $\|c_{ij}\|$, similar to the matrix $\|b_{ij}\|$). It is easy to show that k and $|\det \|b_{ij}\||$ will not depend on the particular choice of the variables $y_1, \dots, y_m$ satisfying the conditions formulated above. Let now

$$q(x_1, \dots, x_m) = \frac{\sqrt{|\det \|b_{ij}\||}}{|\operatorname{grad} L|} \tag{6}$$

be a function of the point of the characteristic cone $x_1, \dots, x_m$, located at unit distance from the origin. Here the matrix $\|b_{ij}\|$ is constructed at the point $x_1, \dots, x_m$, and $\operatorname{grad} L$ is taken at that point on the unit sphere

$$\sum_{i=1}^{m} \omega_i^2 = 1,$$

at which the plane

$$\sum_{i=1}^{m} x_i \omega_i = 0$$

is tangent to the cone $L(\xi_1, \ldots, \xi_m) = 0$.

6. The concrete form of the series (5) depends on the dimension of the space $x_1, \ldots, x_m$ and on the number k, and the coefficient of the first term of this series is equal, up to multiplication by a constant coefficient, to the function $q(x_1, \ldots, x_m)$.

Let m be odd. Then, for even k, the formula holds

$$K(s) = \Phi(s) + \begin{cases} 0, & s > 0; \\ a_1|s|^{n-m/2-1} + a_2|s|^{n-m/2} + \cdots \\ \qquad \cdots + a_k|s|^{n-m/2+k-2} + \cdots, & s < 0. \end{cases}$$

If, however, k is odd, then $K(s)$ is expressed by the formula

$$K(s) = \Phi(s) + \begin{cases} a_1 s^{n-m/2-1} + a_2 s^{n-m/2} + \cdots + a_{ks}^{n-m/2+k-2} + \cdots, & s > 0; \\ 0, & s < 0. \end{cases}$$

Let m be even. Then for $n \geq m/2 + 1$ the following expansions hold:

k even:

$$K(s) = \Phi(s) + [a_1 s^{n-m/2-1} + a_2 s^{n-m/2} + \cdots + a_{ks}^{n-m/2+k-2} + \cdots] \operatorname{sign} s;$$

k odd:

$$K(s) = \Phi(s) + [a_1 s^{n-m/2-1} + a_2 s^{n-m/2} + \cdots + a_{ks}^{n-m/2+k-2} + \cdots] \ln|s|.$$

If, however, m is even and $n < m/2 + 1$, then the following expansions hold:

k even:

$$K(s) = \Phi(s) + a_1\delta^{(m/2-n)}(s) + a_2\delta^{(m/2-n+1)}(s) + \cdots + a_{m/2-n+1}\delta(s) +$$

$$+ [a_{m/2-n+2} + s a_{m/2-n+3} + s^2 a_{m/2-n+4} + \cdots] \operatorname{sign} s;$$

k odd:

$$K(s) = \Phi(s) + a_1 s^{n-m/2-1} + a_2 s^{n-m/2} + \cdots + a_{m/2-n+1} s^{-1} +$$

$$+ [a_{m/2-n+2} + s a_{m/2-n+3} + \cdots] \ln|s|.$$

Recall that in the expansions written above, $\Phi(s)$ is a function analytic in a neighborhood of the value $s = 0$.

The coefficient a_1 is equal to

$$a_1 = c_{m,n,k}\, q(x_1, \dots, x_m),$$

where $q(x_1, \dots, x_m)$ is a function of a point of the characteristic cone and is defined by formula (6), while $c_{m,n,k}$ is a constant coefficient depending on the dimension m of the space, the order n of the equation, and the signature k of the matrix b_{ij}.

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CITED LITERATURE

[1] A. M. Davidova, Candidate dissertation, Moscow State University, 1945. [2] I. M. Gel' fand, Z. Ya. Shapiro, Uspekhi Mat. Nauk, 10, no. 3(65), 3 (1955).

Note: Figure translations are in progress. See original paper for figures.