RELATION BETWEEN THE INVERSION FORMULAS FOR THE SOMMERFELD INTEGRAL AND THE KONTOROVICH-LEBEDEV FORMULAS

The Sommerfeld integral

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Abstract

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RELATION BETWEEN THE INVERSION FORMULAS FOR THE SOMMERFELD INTEGRAL AND THE KONTOROVICH–LEBEDEV FORMULAS

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The Sommerfeld integral

$$S(r,\varphi) = \frac{1}{2\pi i}\int_\gamma e^{-ikr\cos\alpha} s(\varphi+\alpha)\,d\alpha,$$

which gives, in the region $r > 0$, the general solution of the two-dimensional equation of harmonic oscillations $\Delta S + k^2 S = 0$ ($S \sim e^{-i\omega t}$), may, using the symmetry of the contour $\gamma = \gamma_1 + \gamma_2$ (see Fig. 1), be represented in the form

$$S(r,\varphi) = \frac{1}{2\pi i}\int_\gamma e^{-ikr\cos\alpha} \frac{s(\varphi+\alpha) - s(\varphi-\alpha)}{2}\,d\alpha.$$

Considering this integral for a fixed value of φ, we rewrite it briefly as

$$F(r) = \frac{1}{2\pi i}\int_\gamma e^{-ikr\cos\alpha} f(\alpha)\,d\alpha. \qquad (1)$$

If $F(r) = O\{r^{-1+a}\exp(br)\}$ ($a > 0$), then in the class of functions

$$f(\alpha) = O\{\exp[(1-a)|\operatorname{Im}\alpha|]\} \quad (|\operatorname{Im}\alpha| \to \infty)$$

the integral equation (1), where the loops of the contour γ are wholly situated in the regions of regularity of the function $f(\alpha)/\sin\alpha$, has a unique odd solution

$$f(\alpha) = -\frac{ik\sin\alpha}{2}\int_0^\infty F(r) e^{ikr\cos\alpha}\,dr. \qquad (2)$$

Fig. 1

Fig. 1

Figure 1: Fig. 1

Expressions (1), (2) are the inversion formulas for the Sommerfeld integral ([3]), used for solving boundary-value problems in a wedge-shaped domain.

Since the function $f(\alpha)$ is odd, the integral (1) can be written in the form

$$F(r) = \frac{1}{\pi i} \int_{\gamma_1} e^{-ikr\cos\alpha} f(\alpha)\, d\alpha. \tag{3}$$

In the special case when the function $f(\alpha)$ is regular for $|\operatorname{Im}\alpha| > \text{const}$, it follows from (3) that ([4]) $F(0) = 2if(i\infty)$.

We shall show that if $F(0)$, and consequently also $f(i\infty)$, is equal to zero, the transformation of the Sommerfeld integral can be reduced to the Kontorovich-Lebedev transformation ([5]):

$$F(r) = \frac{1}{2} \int_{-i\infty}^{i\infty} e^{-i\nu\pi/2} J_\nu(kr)\, \omega(\nu)\, \nu\, d\nu, \tag{4}$$

$$\omega(\nu) = \int_0^\infty e^{i\nu\pi/2} H_\nu^{(1)}(kr)\, F(r)\, \frac{dr}{r} \; * \, . \tag{5}$$

Since the conditions of applicability of formulas (4), (5) have been investigated in ([5]), we shall restrict ourselves to a formal derivation.

To this end, suppose that the analytic function $f(\alpha)$ is regular in the strip $|\operatorname{Re}\alpha| < \pi/2 + \varepsilon$ and decreases there as $|\operatorname{Im}\alpha| \to \infty$ like $f(\alpha) = O(e^{-c|\operatorname{Im}\alpha|})$. Then we represent the function $f(\alpha)$ by a Fourier integral of the form ([4])

$$f(\alpha) = \frac{i}{2} \int_{-i\infty}^{i\infty} g(\nu) e^{-i\nu\alpha}\, d\nu, \tag{6}$$

absolutely convergent in this strip, where the function

$$g(\nu) = \frac{i}{\pi} \int_{-i\infty}^{i\infty} f(\alpha) e^{i\nu\alpha}\, d\alpha \tag{7}$$

is odd, just as $f(\alpha)$ is, and moreover the last integral is absolutely convergent in the strip $|\operatorname{Re}\nu| < c$.

Substituting (6) into (3), we have

$$F(r) = -\frac{1}{2\pi}\int_{\gamma_1} d\alpha \int_{-i\infty}^{i\infty} g(\nu) e^{-i(kr\cos\alpha - \nu\alpha)}\, d\nu, \tag{8}$$

where we have changed the sign of the integration variable ν to the opposite one and used the oddness of $g(\nu)$.

Regarding temporarily the quantity k as positive imaginary, we can deform the contour γ_1 so that it lies entirely in the strip $|\operatorname{Re}\alpha| < \pi/2 + \varepsilon$. In this case integral (8) becomes absolutely convergent, and the interchange of the order of integration is legitimate.

In the integral

$$F(r) = -\frac{1}{2\pi}\int_{-i\infty}^{i\infty} g(\nu)\, d\nu \int_{\gamma_1} e^{-i(kr\cos\alpha - \nu\alpha)}\, d\alpha$$

one may return to positive values of the parameter k and to the original form of the contour.

Hence, since

$$J_\nu(kr) = -\frac{\exp(i\nu\pi/2)}{2\pi}\int_{\gamma_1} e^{-i(kr\cos\alpha - \nu\alpha)}\, d\alpha,$$

we obtain

$$F(r) = \int_{-i\infty}^{i\infty} e^{-i\nu\pi/2} J_\nu(kr) g(\nu)\, d\nu. \tag{9}$$

To obtain the inverse formula, splitting integral (2) into parts:

$$\int_0^\infty = \int_0^a + \int_a^\infty,$$

and integral (7) into parts

$$\int_{-i\infty}^{i\infty} = \int_{-ib}^{ib} + \left(\int_{-i\infty}^{-ib} + \int_{ib}^{i\infty}\right)$$

and substituting (2) into (7), we have

* The authors use $H_\nu^{(2)}(kr)$.

$$g(\nu) = -\frac{k}{2\pi}\left(\int_{-i\infty}^{-ib} + \int_{ib}^{i\infty}\right) d\alpha \int_{a}^{\infty} F(r) e^{i(kr\cos\alpha + \nu\alpha)} \sin\alpha \, dr + A,$$

where by A are denoted integrals tending to zero as $a, b \to 0$. The first term on the right-hand side for $\operatorname{Im} k > 0$ is an absolutely convergent integral, in which the order of integration may be interchanged.

Therefore, in the limit we obtain

$$g(\nu) = -\frac{k}{2\pi}\int_{0}^{\infty} F(r)\, dr \int_{-i\infty}^{i\infty} e^{i(kr\cos\alpha + \nu\alpha)} \sin\alpha \, d\alpha,$$

or, since

$$\int_{-i\infty}^{i\infty} e^{i(kr\cos\alpha + \nu\alpha)} \sin\alpha \, d\alpha = -\frac{\pi\nu}{kr} e^{i\nu\pi/2} H_{\nu}^{(1)}(kr),$$

$$g(\nu) = \frac{\nu}{2} e^{i\nu\pi/2} \int_{0}^{\infty} F(r) H_{\nu}^{(1)} kr \, \frac{dr}{r}. \tag{10}$$

If in (9) and (10), instead of the odd function $g(\nu)$, we introduce the even function $\omega(\nu) = \frac{2}{\nu} g(\nu)$, we obtain the Kontorovich-Lebedev transformation formulas (4), (5).

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Note: Figure translations are in progress. See original paper for figures.