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Abstract

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THE RESIDUE METHOD FOR SOLVING MIXED PROBLEMS AND SOME RELATED EXPANSION FORMULAS

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In this note a formula is given that represents a sufficiently smooth solution of mixed problems for a system of linear differential equations with piecewise-smooth coefficients under boundary conditions that relate derivatives of the unknown vector function at more than two points of the interval of variation of the spatial variable x and that also contain derivatives of the unknown vector function with respect to time, in the form of an integral residue (see formula (8)). In connection with this, a new expansion formula is also given (see formula (6)) for arbitrary vector functions in series in the fundamental functions of the spectral problem corresponding to problem (1)–(3). The results of this note generalize part of the results of the work [1].

The proposed method is also applicable to some problems encountered in applications for which the spectral problems do not correspond to self-adjoint operators (see examples 1 and 2).

1. Let us consider the problem of finding the solution of the system

$$\frac{\partial u_j^{(i)}}{\partial t} = u_{j+1}^i \qquad (j = 0, \dots, q-2),$$

$$\frac{\partial u_{q-1}^{(i)}}{\partial t} = \sum_{\substack{k<q-1 \\ mk+l\leq p}} A_{kl}^{(i)}(x) \frac{\partial^l u_k^{(i)}}{\partial x^l} + f^{(i)}(x,t), \qquad x \in (a_i, b_i) \quad (i = 1, \dots, n) \quad (1)$$

under the boundary conditions

$$\sum_{i=1}^{n} \sum_{\substack{l\leq p-1 \\ k\leq q-1}} \left\{ \alpha_{kl}^{(i)} \left. \frac{\partial^l u_k^{(i)}}{\partial x^l} \right|_{x=a_i} + \beta_{kl}^{(i)} \left. \frac{\partial^l u_k^{(i)}}{\partial x^l} \right|_{x=b_i} \right\} +$$

$$+\sum_{i=1}^{n}\sum_{l=0}^{p-1}\left\{\alpha_{ql}^{(i)}\left.\frac{\partial^{l+1}u_k^{(i)}}{\partial t\,\partial x^l}\right|_{x=a_i}+\beta_{kl}^{(i)}\left.\frac{\partial^{l+1}u_k^{(i)}}{\partial t\,\partial x^l}\right|_{x=b_i}\right\}=0 \tag{2}$$

and the initial conditions

$$u_k^{(i)}(x,0)=\varphi_k^{(i)}(x)\quad \text{for } x\in(a_i,b_i), \tag{3}$$

where $A_{kl}^{(i)}(x)$ are square matrices of order r; $\alpha_{kl}^{(i)},\beta_{kl}^{(i)}$ are constant matrices of size $nrp\times p$; $f^{(i)}(x,t)$, $\varphi_k^{(i)}(x)$ are vector functions of the corresponding dimension; $p=mq$; (a_i,b_i) are mutually non-overlapping intervals having common endpoints. The mixed problem obtained from (1)–(3) for the vector functions $u_0^{(i)}$ shall be called the **basic mixed problem**.

2. We shall call the problem of finding a solution of the system

$$v_{j+1}^{(i)}-\lambda^m v_j^{(i)}=\varphi_j^{(i)}(x)\quad (j=0,\dots,q-2),$$

$$\sum_{\substack{mk+l\le p\\ k\le q-1}}A_{kl}^{(i)}(x)\frac{d^l v_k^{(i)}}{dx^l}-\lambda^m v_{q-1}^{(i)}=\varphi_{q-1}^{(i)}(x)\quad \text{for } x\in(a_i,b_i) \tag{4}$$

under the boundary conditions

$$\sum_{i=1}^{n}\sum_{\substack{l\le p-1\\ k\le q-1}}\left\{\alpha_{kl}^{(i)}\left.\frac{d^l v_k^{(i)}}{dx^l}\right|_{x=a_i}+\beta_{kl}^{(i)}\left.\frac{d^l v_k^{(i)}}{dx^l}\right|_{x=b_i}\right\}+\lambda^m\sum_{i=1}^{n}\sum_{l=0}^{p-1}\left\{\alpha_{ql}^{(i)}\left.\frac{d^l v_{q-1}^{(i)}}{dx^l}\right|_{x=a_i}+\beta_{ql}^{(i)}\left.\frac{d^l v_{q-1}^{(i)}}{dx^l}\right|_{x=b_i}\right\}=0 \tag{5}$$

the spectral problem corresponding to problem (1)–(3).

Fix s among the numbers $0,1,\dots,q-1$. Obviously, from the first $q-1$ equations of system (4), expressing all $v_k^{(i)}$ in terms of $v_s^{(i)}$ and substituting these expressions into the last of equations (4) and into the boundary condition (5), we arrive at a problem with the unknown vector-function $v_s^{(i)}$. We shall call this problem the s-auxiliary spectral problem. The inhomogeneous term of the system of the s-auxiliary spectral problem will be denoted by $F_s(x,\varphi,\lambda^m)$, and the inhomogeneous term of the boundary conditions by $N_s(\varphi,\lambda^m)$. In what follows we shall call the 0-auxiliary problem the spectral problem corresponding to the principal mixed problem.

Let the following conditions be satisfied:

1°. For $x \in [a_i, b_i]$ $(i = 1, \dots, n)$, $A_{kl}^{(i)}(x)$ are continuous, twice continuously differentiable when $mk + l = p$, continuously differentiable when $mk + l = p - 1$, $\det A_{0p}^{(i)}(x) \neq 0$, and $\varphi_{k-1}^{(i)}(x)$ are continuously differentiable $p - mk$ times for $k = 1, \dots, q$.

2°. For $x \in [a_i, b_i]$ the roots $\psi_1^{(i)}(x), \dots, \psi_{rp}^{(i)}(x)$ of the characteristic equations

$$\det\left(\theta^p A_{0,p}^{(i)}(x) + \theta^{p-m} A_{q-1,m}^{(i)}(x) + \dots + \theta^m A_{1,(q-1)m}^{(i)}(x) - E\right) = 0$$

are distinct and different from zero; their arguments and the arguments of their differences do not depend on x. For large λ, the sign of the differences $\operatorname{Re} \lambda \varphi_k^i(x) - \operatorname{Re} \varphi_j^i(x)$ does not depend on x.

3°. For sufficiently large λ,

$$\operatorname{rank}(\alpha_0^{(1)}(\lambda), \dots, \alpha_{p-1}^{(1)}(\lambda), \dots, \alpha_0^{(n)}(\lambda), \dots, \alpha_{p-1}^{(n)}(\lambda), \beta_0^{(1)}(\lambda), \dots, \beta_{p-1}^{(n)}(\lambda)) = npr,$$

where

$$\alpha_j^{(s)}(\lambda) = \sum_{k=0}^{q} \lambda^{mk+j} \alpha_{kj}^{(s)}, \qquad \beta_j^{(s)}(\lambda) = \sum_{k=0}^{q} \lambda^{mk+j} \beta_{kj}^{(s)}.$$

4°. For large λ and for every $s = 0, \dots, q-1$, the estimate

$$\lambda^{m(s+1)} N_s(\varphi, \lambda^m) = O(1)$$

holds.

It is easy to verify that by the substitution

$$\lambda^{-l} \frac{d^l v_s^{(i)}}{dx^l} = w_{sl}^{(i)} \quad (l = 0, \dots, p-1)$$

the s-auxiliary spectral problem can be reduced to a problem with a parameter for a system of first-order equations. Let $\Delta(\lambda)$ be the characteristic determinant of the Green' s matrix of this problem. Using the method indicated in note ([2]), the λ-plane can be divided into sectors (T_j), in which, for large λ, the asymptotic formula holds

$$\Delta(\lambda) = \lambda^l \exp\left[\lambda \sum_{k,i} W_k^{(i,2)}\right] \{[M_{1j}] \exp[m_{1j} z] + \dots + [M_{\sigma j}] \exp[m_{\sigma j} z],$$

in which the notation of note ([2]) is preserved.

In addition to conditions 1°–4°, we shall assume:

5°. All the numbers $M_{1j}, M_{\sigma j}$ for the boundary-value problem with unknowns $w_{sl}^{(i)}$ are different from zero.

Let us further note that, according to conditions 3°–4°, the boundary-value problem for $w_{sl}^{(i)}$ can be reduced to a problem for a system of linear differential equations of first order with homogeneous boundary conditions. According to conditions 1°–5°, for this new problem all the conditions of the theorem of note ([2]) are fulfilled. Therefore, by applying the theorem mentioned, one easily proves:

Theorem 1. *Under conditions 1°–5°, the expansion formula holds*

$$\frac{1}{2\pi\sqrt{-1}} \sum_{\nu} \int_{c_\nu} \lambda^{m-1} v_s^{(i)}(x, \varphi, \lambda^m)\, d\lambda = \varphi_s^{(i)}(x) \tag{6}$$

$$(s = 0, \dots, q-1;\ i = 1, \dots, n),$$

where c_ν is a closed contour surrounding only one pole λ_ν of the function $v_s^{(i)}(x, \varphi, \lambda^m)$ (the solution of the s-auxiliary spectral problem), and summation over ν extends over all poles of this function; convergence of the series is understood in the sense of $L_2(a_i, b_i)^*$.

Expressing $v_s^{(i)}$ through $v_0^{(i)}$ from formula (6), we directly obtain the expansion formula

$$-\frac{1}{2\pi\sqrt{-1}} \sum_{\nu} \int_{c_\nu} \lambda^{m(s+1)-1} \sum_{j=1}^{n} \int_{a_j}^{b_j} G^{(i,j)}(x, \xi, \lambda)\, \varphi_{q-1}^{(j)}(\xi)\, d\xi = \begin{cases} 0, & \text{for } s < q-1, \\ \varphi_{q-1}^{(i)}(x), & \text{for } s = q-1, \end{cases} \tag{7}$$

where $G^{(i,j)}(x, \xi, \lambda)$ is the Green' s matrix of the spectral problem corresponding to the original mixed problem.

3. By the method of work ([1]) one proves:

Theorem 2. *Suppose that, under the conditions of Theorem 1, the boundary condition of the original mixed problem does not contain a derivative with respect to t of order q, and this problem, for $f^{(i)}(x,t) \in L_2(a_i, b_i)$,*

$$\int_0^t \int_{a_i}^{b_i} |f_k^{(i)}(x,t)|\, dx\, dt < \infty$$

has a solution $u_0^{(i)}(x,t)$ satisfying condition 4° of Theorem 1 and possessing the properties:

1°. $u_0^{(i)}, \dots, \partial^{q-1}u_0^{(i)}/\partial t^{q-1}$ have derivatives with respect to x in (a_i, b_i) up to order p; the derivatives of order $(p-1)$ are absolutely continuous with respect to $x \in (a_i, b_i)$, and the derivatives of order p belong to $L_2(a_i, b_i)$.

* By imposing certain restrictions on φ, one can obtain convergence in the pointwise sense.

2°. The derivatives $\partial^{k+l}u_0^{(i)}/\partial t^k \partial x^l$ $(k = 0, \dots, q-1;\ l = 0, \dots, p-1)$ are absolutely continuous in t on $[0,T]$, and the derivatives with respect to t of these derivatives are summable in the two-dimensional domain $0 \le t \le T,\ a_i < x < b_i$.

Then this solution is represented by the formula

$$u_0^{(i)}(x,t) = -\frac{1}{2\pi\sqrt{-1}} \sum_{\nu} \int_{C_\nu} \lambda^{m-1} e^{\lambda^m t}\, d\lambda \left\{ \sum_{j=1}^{n} \int_{a_j}^{b_j} G^{(i,j)}(x,\xi,\lambda) \left[F_0^{(j)}(\xi,\varphi,\lambda^m) + \int_0^t e^{-\lambda^m \tau} f^{(j)}(\xi,\tau)\, d\tau \right] d\xi + \right. \tag{8}$$

where $\Delta^{(i)}(x,\varphi,\lambda^m)$ is the solution of the 0-auxiliary spectral problem for the corresponding homogeneous system.

Example 1. In underground hydromechanics, in connection with the study of the hydrodynamic parameters of a bed, the following mixed problem arises:

$$\frac{\partial^2 u}{\partial t^2} + \delta c^2 \frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} \right), \qquad \left.\frac{\partial u}{\partial r}\right|_{r=a} + \alpha \left.\frac{\partial u}{\partial t}\right|_{r=a} = 0,$$

$$u(b,t) = 0, \qquad \left.\frac{\partial^k u}{\partial t^k}\right|_{t=0} = \varphi_k(r) \quad (k = 0, 1),$$

where δ, c, α, β are constants.

The corresponding problem, taking into account the smallness of $1/c^2$, is solved in the book by I. A. Charny [3] for the heat-conduction equation. If $\varphi_1(r)$ is discontinuous, $\varphi_0(r)$ is twice continuously differentiable on $[a,b]$, and $\varphi_0(b) = 0$, then for this problem all the conditions of Theorem 2 are satisfied. In this case $m = 1$, $F_0(\rho,\varphi,\lambda) = \varphi_1(\rho) + (\lambda + \delta c^2)\varphi_0(\rho)$.

Example 2. Consider the problem of heat propagation in a nonhomogeneous medium:

$$\frac{\partial u_i}{\partial t} = \chi_i \left(\frac{\partial^2 u_i}{\partial r^2} + \frac{1}{r}\frac{\partial u_i}{\partial r} \right) + f_i(r,t) \qquad \text{for } r \in (a_{i-1}, a_i)$$

under the boundary conditions

$$\left.\frac{\partial u_i}{\partial r}\right|_{r=a_i} + h_i\{u_{i+1}(a_i, t) - u_i(a_i, t)\} = 0,$$

$$\left.\frac{\partial u_i}{\partial r}\right|_{r=a_{i-1}} + H_i\{u_i(a_{i-1}, t) - u_{i-1}(a_{i-1}, t)\} = 0$$

and the initial conditions

$$u_i(r, 0) = \varphi_i(r) \qquad \text{for } r \in (a_{i-1}, a_i) \quad (i = 1, 2, 3).$$

For smooth $f_k(r, t)$, $\varphi_k(r)$, the conditions of Theorem 2 are also satisfied for this problem.

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CITED LITERATURE

[1] M. L. Rasulov, *Matem. sborn.*, **30** (72), 3 (1952).
[2] M. L. Rasulov, *DAN*, **119**, No. 3 (1958).
[3] I. A. Charny, *Underground Hydromechanics*, 1948.

Note: Figure translations are in progress. See original paper for figures.