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Abstract

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S. N. SLUGIN

A MODIFICATION OF THE ABSTRACT ANALOGUE OF CHAPLYGIN' S METHOD

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1. In papers ([1,2]), for the approximate solution of operator equations in semi-ordered spaces ([3]), an analogue (which we shall call abstract) of Chaplygin' s method ([4]) was proposed. But the actual execution of the indicated algorithms may lead to cumbersome expressions that make further computations difficult. In the present paper one of the ways of constructing simplified algorithms is indicated, giving approximations in the Chaplygin sense.

2. We use here the concepts and propositions given in ([3]). The operator P maps X into Y; X and Y are K-spaces. We shall call an operator Γ **positively invertible** if it has an inverse $\Gamma^{-1} > 0$. The operators Γ and Λ will be called, respectively, a **majorant** and a **minorant** on $[a, b]$ of the increment ΔP, if

$$\Gamma(\Delta x) \geq P(x + \Delta x) - P(x) \geq \Lambda(\Delta x),$$

where $[x, x + \Delta x] \subseteq [a, b]$. If the operators H_1 and H_2 are defined for $x \leq 0$, then $H_1 \geq H_2$ will mean that $H_1(x) \geq H_2(x)$ for all $x \leq 0$; the n-th iteration of an operator V will be denoted by V^n, and the identity operator by I.

Theorem 1. 1) Let on each $[a, b] \subseteq [\underline{x}_0, \bar{x}_0]$ there exist additive positively invertible majorants $\Gamma(a, b)$ of the increment ΔP, with

$$\Gamma(a, b) \leq \Gamma(\underline{x}_0, \bar{x}_0) = \Gamma_0,$$

and let the operators Γ_0^{-1} and P be monotone-continuous, and

$$P(\underline{x}_0) \leq 0 \leq P(\bar{x}_0).$$

Then the algorithm

$$\underline{x}_{n+1} = \underline{x}_n - \Gamma_n^{-1}(\underline{z}_n), \qquad \bar{x}_{n+1} = \bar{x}_n - \Gamma_n^{-1}(\bar{z}_n) \quad (n = 0, 1, 2, ...)$$

(z_n are arbitrary elements satisfying the inequalities

$$P(\underline{x}_n) \leq \underline{z}_n \leq 0 \leq \bar{z}_n \leq P(\bar{x}_n);$$

$$\Gamma(\underline{x}_n, \bar{x}_n) = \Gamma_n$$

) defines sequences x_n such that

$$\underline{x}_n \le \underline{x}_{n+1} \le \underline{x} \le \bar{x} \le \bar{x}_{n+1} \le \bar{x}_n, \qquad (1)$$

where $\underline{x}, \bar{x}$ are the least and greatest solutions on $[\underline{x}_0, \bar{x}_0]$ of the equation

$$P(x) = 0.$$

2) If

$$\underline{z}_n = \underline{H}_n P(\underline{x}_n), \qquad \bar{z}_n = \bar{H}_n P(\bar{x}_n),$$

and 3) if there exists a monotone-continuous positive operator $H \le \underline{H}_n$, $H \le \bar{H}_n$, then $\underline{x}_n \uparrow \underline{x}$, $\bar{x}_n \downarrow \bar{x}$.

If there exists an additive positively invertible minorant Λ of the increment ΔP, then the solution is unique.

Instead of invertibility of Λ, one may require that

$$(I - \Gamma_0^{-1}\Lambda)^n(\bar{x}_0 - \underline{x}_0) \to 0 \quad \text{as } n \to \infty.$$

The initial approximations may be sought in the form

$$\underline{x}_0 = u - \Lambda^{-1}(z), \qquad \bar{x}_0 = v - \Lambda^{-1}(\bar{z}),$$

where $u \le v$ are arbitrary elements of the space X; $z, \bar{z}$ are elements of Y satisfying the inequalities $z \ge |P(u)|$, $\bar{z} \le -|P(v)|$.

The approximations $\underline{x}_0, \bar{x}_0$ may also be sought in the form

$$\underline{x}_0 = L^{-1}(z), \qquad \bar{x}_0 = L^{-1}(\bar{z}), \qquad z \le \inf(L-P)(x), \qquad \sup(L-P)(x) \le \bar{z},$$

where L is an arbitrary operator having a monotone increasing inverse (for example, L^{-1} is additively positive).

The theorem can be proved, for example, on the basis of Theorem 1 ([2]), in which, for $\Gamma_n = \bar{\Gamma}_n$, the algorithm $x_{n+1} = x_n - \Gamma_n^{-1}P(x_n)$ is given. Introducing z_n instead of $P(x_n)$, although it gives cruder approximations in comparison with this algorithm, makes it possible to simplify the computations and thereby to construct better approximations by allowing a greater number of practically possible steps of the process. The elements z_n can sometimes be chosen even without computing the exact value of $P(x_n)$, by determining only its approximation with an accuracy sufficient for the given step of the process (see the examples).

Theorem 2. *If condition 1) of Theorem 1 is satisfied, the algorithm*

$$\underline{x}_{n+1} = \underline{x}_n - z_n, \qquad \bar{x}_{n+1} = \bar{x}_n - \bar{z}_n$$

(where z_n are arbitrary elements satisfying the inequalities $\Gamma_0^{-1} P(\underline{x}_n) \leq z_n \leq 0 \leq \bar{z}_n \leq \Gamma_0^{-1} P(\bar{x}_n)$) determines the sequences (1).

If $z_n = H_n \Gamma_0^{-1} P(\underline{x}_n)$, $\bar{z}_n = \bar{H}_n \Gamma_0^{-1} P(\bar{x}_n)$ and condition 2) is satisfied, then $\underline{x}_n \uparrow x$, $\bar{x}_n \downarrow \bar{x}$.

The sufficient conditions for uniqueness are the same as before.

The elements $x_0, \bar{x}_0$ may satisfy the weaker conditions: $\Gamma_0^{-1} P(x_0) \leq 0 \leq \Gamma_0^{-1} P(\bar{x}_0)$. To prove Theorem 2 it suffices in Theorem 1 to put all $\Gamma_n = I$ and to take $\Gamma_0^{-1} P$ instead of P.

The processes of the theorems may be combined, beginning with the algorithm of Theorem 1 and continuing with the algorithm of Theorem 2. In this transition one may use for one step the formula $x_{n+1} = x_n - K(z_n)$, where the arbitrary positive operator $K \leq \Gamma_n^{-1}$ is simpler than Γ_n^{-1}.

3. In the paper ([5]) Chaplygin' s method is applied to the boundary-value problem

$$y'' - f(t, y) = 0, \qquad y(a) = \alpha, \qquad y(b) = \beta, \qquad 0 \leq \partial f / \partial y \leq A.$$

Theorems 1 and 2 may also be applied to these equations; moreover, the condition $\partial f / \partial y \geq 0$ may be replaced by the condition $\partial f / \partial y \geq -B$ ($B \geq 0$, with an upper bound specified).

Put

$$P(y) = f(t, y) - y'', \qquad a \leq t \leq b, \qquad \Gamma_0(\Delta y) = A \Delta y - \Delta y'',$$

$$\Lambda(\Delta y) = -B \Delta y - \Delta y'', \qquad \Delta y(a) = \Delta y(b) = 0,$$

and let the functions $y_0, \bar{y}_0$ satisfy the given boundary conditions.

If $\Gamma_0(\Delta y) = \varphi(t) \geq 0$, then

$$\Delta y = \Gamma_0^{-1}(\varphi) = -\int_a^b G(t, \tau) \varphi(\tau) \, d\tau \geq 0,$$

since the Green' s function $G(t, \tau) \leq 0$ in the present case. The positivity of Λ^{-1} is established in the same way in the case $B \leq \pi^2 / (b - a)^2$.

Example. $y^2 + 2 - y'' = 0$, $y(0) = y(1) = 0$. Taking $y_0 = 2(t^2 - t)$, $\bar{y}_0 \equiv 0$, we find $A = 0$, $B = 1$. The algorithm of Theorem 1 here takes the form

$$(\Gamma_n = \Gamma_0): \quad y_{n+1} = y_n + (t-1)\int_0^1 \tau z_n \, d\tau + t\int_t^1 (\tau - 1) z_n \, d\tau.$$

Putting $z_0 = -7/4$, $\overline{z}_0 = P(\overline{y}_0)$, $z_1 = P(y_1)$, we establish that $\overline{y}_1 - y_1 \leq 1/32$ and

$$y_2 = t^2 - t - \frac{81}{64}J(t), \qquad \overline{y}_2 = t^2 - t - J(t),$$

$$J(t) = \frac{1}{60}t(1 - 5t^3 + 6t^5 - 2t^6), \qquad \overline{y}_2 - y_2 < 0.01.$$

Instead of $\Gamma_0^{-1}(z_2)$ one may take

$$K(z_2) = (1-t)\int_0^t \tau z_2 \, d\tau$$

and pass to the process of Theorem 2.

4.

$$f_1 \equiv 2t_1^2 - 2t_1t_3 + t_3^2 + 4t_1 - 2 = 0,$$
$$f_2 \equiv t_1^2 + t_2^2 - 2t_1 + 8t_2 + 1 = 0,$$
$$f_3 \equiv t_2^2 + t_3^2 - 2t_2 + 10t_3 + 2 = 0.$$

Put $x_0 = (0, -1, -1)$, $\overline{x}_0 = (1, 1, 0)$. As Γ_0 and Λ we take ([6]) linear transformations with matrices $\|a_{ik}\|$, $\|b_{ik}\|$, where $a_{ik} = \max \partial f_i/\partial t_k$, $b_{ik} = \min \partial f_i/\partial t_k$ in the indicated parallelepiped $[x_0, \overline{x}_0]$. We compute $\Gamma_0^{-1}P(x_n)$ for $n = 0, 1, 2$ to tenths, for $n = 3, \ldots, 7$ to hundredths, and for $n = 8$ to thousandths. Simple calculations lead to the result

$$x_9 = (0.369; -0.054; -0.218), \qquad \overline{x}_9 = (0.383; -0.045; -0.212).$$

5. In conclusion we shall demonstrate the application of Theorems 1 and 2 to the delay equation

$$y'(t) - \frac{1}{2}y^3\left(\frac{t}{2}\right) - 1 = 0, \qquad y(0) = 0, \qquad 0 \leq t \leq 1.$$

Putting $y_0 \equiv 0$, $\overline{y}_0 = 2t$, we establish that $\Gamma_0 = I$, $\Lambda^{-1} > 0$ in the space of derivatives $y'(t)$ ([2]). Put $z_0 = P(y_0)$ and then, estimating the higher powers of t, we find

$$P(\overline{y}_1) = \frac{1}{2}t^3 - \frac{t^2}{16}\left(1 + \frac{t^3}{64}\right)^3 = \frac{7}{16}t^3 - \frac{3t^6}{2^{10}} - \cdots \geq \frac{7}{16}t^3 - \frac{t^6}{2^8} = \overline{z}_1, \qquad z_1 = P(y_1).$$

Computing similarly $P(y_2)$ to accuracy up to t^6 and $P(y_3)$ to t^9, we obtain from the formula $y'_{n+1} = y'_n - P(y_n)$

$$y_4 = t + \frac{t^4}{64} + \frac{3t^7}{7 \cdot 2^{13}} + \frac{9t^{10}}{35 \cdot 2^{24}}, \qquad \overline{y}_4 = y_4 + \frac{t^{10}}{5 \cdot 2^{24}}.$$

The process is easily continued.

6. For the algorithm of Theorem 1 the following estimate of the rate of convergence is valid (if H is additive):

$$|\overline{x}_n - x_n| \leq (I - \Gamma_0^{-1} H \Lambda)^n (\overline{x}_0 - x_0).$$

After substituting $H\Gamma_0^{-1}$ in place of $\Gamma_0^{-1}H$, we obtain the estimate for the algorithm of Theorem 2.

If only the composite operator $\Gamma_0^{-1}P$ is continuous ([2]), then, for convergence of the algorithm of Theorem 1, instead of the continuity of H it suffices to require additivity and invertibility of H and monotone continuity of the operator $\Gamma_0^{-1}HP$.

Practically, the best criterion of convergence to the solution remains, as in any Chaplygin-type method, the direct estimate of the difference $\overline{x}_n - x_n$ (if the solution is unique).

Komsomolsk-on-Amur State
Pedagogical Institute

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Note: Figure translations are in progress. See original paper for figures.