Reports of the Academy of Sciences of the USSR

Yu. P. KRIVENKOV

1958

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1958. Vol. 123, No. 2

MATHEMATICS

Yu. P. KRIVENKOV

REPRESENTATIONS OF SOLUTIONS OF THE EULER-POISSON-DARBOUX EQUATION WITH NEGATIVE COEFFICIENT

(Presented by Academician I. N. Vekua on 30 V 1958)

In papers ([1,2]), representations by means of analytic functions of one complex variable were studied for solutions of the equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{c}{y}\frac{\partial w}{\partial y} = 0, \qquad c = \text{const} \tag{1}$$

for values $c > 0$. In the present paper, for this same equation, representations by means of analytic functions of solutions are investigated for $c < 0$.

Let us denote by T a simply connected domain lying in the half-plane $y > 0$ and adjoining an interval ab (or L) of the axis Ox, and by $\overline{T}$ the domain symmetric to it with respect to the axis Ox. We shall say that the domain T (or $\overline{T}$) **belongs to class** B if the domain $T \cup L \cup \overline{T}$ contains entirely the segment joining any two of its points that have the same abscissas.

Consider the classes $E_\alpha(T)$ and $N_\alpha(T)$ of solutions of equation (1). To the class $E_\alpha(T)$ we assign every function continuous in $T \cup L$ and twice continuously differentiable in T, satisfying equation (1) for $c = \alpha$. To the class $N_\alpha(T)$ we assign those functions of the class $E_\alpha(T)$ which on L satisfy the condition

$$\lim_{y \to 0} y^\alpha \frac{\partial w}{\partial y} = 0.$$

Correspondingly, we consider the classes $E_\alpha(\overline{T})$ and $N_\alpha(\overline{T})$ of solutions defined in $\overline{T}$.

Lemma 1. *Every solution $w(x,y) \in E_c(T)$ $(c < 0)$, satisfying on L the condition $w(x,0) = 0$, in any closed domain $\widetilde{T}' \subset T \cup L$ adjoining L, can be represented in the form*

$$w(x,y) = yA(x,y), \tag{2}$$

where $A(x,y)$ is bounded in $\widetilde{T}'$.

Proof. For an arbitrary point $(x_0, 0)$ belonging to L, construct the barrier function

$$v(x,y) = y\left[1 - \alpha e^{-p/(x-x_0)^2}\right] + \beta(x-x_0)^2.$$

Choose α, β, and p so that in T' one has $v(x,y) > 0$ and

$$\Delta v + \frac{c}{y} v'_y = -\frac{2\alpha p y}{(x-x_0)^6} e^{-p/(x-x_0)^2}\left[2p - 3(x-x_0)^2\right] - \frac{-c - 2\beta y + c\alpha e^{-p/(x-x_0)^2}}{y} < 0.$$

Consider the functions $\varphi(x,y) = kv - w$ and $\psi(x,y) = -kv - w$; by the usual arguments we find that in T', $\varphi(x,y) \geq 0$ and $\psi(x,y) \leq 0$. Formula (2) follows from the latter expressions.

Theorem 1. *If a solution $w(x,y) \in E_c(T)$ $(-2 < c < 0)$ satisfies on L the condition $w(x,0) = 0$, then it is represented in T in the form*

$$w(x,y) = \gamma\left(1 - \frac{c}{2}\right)\left(\frac{y}{1-c}\right)^{1-c} \int_0^1 \frac{\psi[x + iy(1-2\sigma)]\, d\sigma}{[\sigma(1-\sigma)]^{c/2}}, \tag{3}$$

where $\gamma(\alpha) = \Gamma(2\alpha)/\Gamma^2(\alpha)$ and $\psi(z)$ is a function analytic in T and satisfying on L the condition

$$\lim_{y\to 0}\left(\frac{y}{1-c}\right)^c \frac{\partial w}{\partial y} = \psi(x). \tag{4}$$

Proof. In a neighborhood T' of the interval L, such that the domain $T' \cup L \cup \overline{T'}$ is convex, consider the expression

$$v(x,y) = \frac{M}{c}\left[\int_{y_0}^{y} y^c w(x,y)\, dy - y_0^c \int_{x_0}^{x}\int_{x_0}^{x_1} w'_y(x_2, y_0)\, dx_2 dx_1\right],$$

in which (x_0, y_0) is some point of T', and

$$M = 4\frac{\gamma(-c/2)}{\gamma(1-c/2)}(1-c)^{1-c}.$$

We shall show that $v(x,y) \in N_{-c}(T')$. By virtue of Lemma 1, $w(x,y) = yA(x,y)$, and therefore $v(x,y)$ is continuous on L. Moreover, $v(x,y)$ satisfies in T' equation (1) with coefficient c equal to $-c$. Indeed:

$$\frac{c}{M}\left(v''_{x^2} + v''_{y^2} - \frac{c}{y}v'_y\right) = \int_{y_0}^{y} y^c w''_{x^2}\,dy + \int_{y_0}^{y} \frac{d}{dy}(y^c w'_y) =$$

$$= \int_{y_0}^{y} y^c \left(w''_{x^2} + w''_{y^2} + \frac{c}{y}w'_y\right) dy.$$

Next

$$\lim_{y\to 0} y^{-c}\frac{\partial v}{\partial y} = w(x,0) = 0.$$

Therefore $v(x,y)$ is represented in T' (see ([1],[2])) in the form

$$\Pi_{-c/2}\Phi(z) = \gamma\left(-\frac{c}{2}\right)\int_0^1 \frac{\Phi[x+iy(1-2\sigma)]\,d\sigma}{[\sigma(1-\sigma)]^{1+c/2}},$$

where $\Phi(z)$ is a function analytic in T'. We perform the following transformations:

$$My^{-1+c}w(x,y) = \frac{c}{y}\frac{\partial}{\partial y}\Pi_{-c/2}\Phi(z) - \left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}\right)\Pi_{-c/2}\Phi(z) =$$

$$= \gamma\left(-\frac{c}{2}\right)4\frac{\partial^2}{\partial z\,\partial\zeta}\int_0^1 \frac{\Phi[z(1-\sigma)+\zeta\sigma]\,d\sigma}{[\sigma(1-\sigma)]^{1+c/2}}\Bigg|_{\zeta=\bar{z}} = 4\frac{\gamma(-c/2)}{\gamma(1-c/2)}\Pi_{-c/2+1}\Phi''(z) =$$

$$= M(1-c)^{c-1}\Pi_{-c/2+1}\psi(z),$$

where $\psi(z) = \Phi''(z)$.

We obtain that in T' representation (3) is valid. Applying H. Heinrich's theorem II ([3]) to the expression

$$w(x,y)\left(\frac{y}{1-c}\right)^{c-1} = \gamma\left(1-\frac{c}{2}\right)\int_0^1 \frac{\psi[x+iy(1-2\sigma)]\,d\sigma}{[\sigma(1-\sigma)]^{c/2}},$$

which is analytic in the domain $T \cup L \cup \overline{T}$, we obtain that $\psi(z)$ is analytic in T, and therefore $w(x,y)$ is represented in the form (3) throughout the whole domain T.

We shall denote the classes $E_\alpha(T)$ and $N_\alpha(T)$ with an asterisk if, for solutions of these classes, the condition

$$\frac{\partial^2 w}{\partial x^2} = O(y^{-1-c+\varepsilon}), \qquad \varepsilon > 0. \tag{5}$$

Lemma 2. Any solution $w(x,y) \in N_c^*(T)$ $(-1 < c < 0,\ T \in B)$ is represented in T in the form

$$w(x,y) = \frac{1}{1+c}\, y^{-c} \frac{\partial}{\partial y}\, y^{c+1} v_0(x,y),$$

where

$$v_0(x,y) = \gamma \left(1 + \frac{c}{2}\right) \int_0^1 \frac{\varphi[x + iy(1-2\sigma)]\, d\sigma}{[\sigma(1-\sigma)]^{-c/2}}, \tag{6}$$

and $\varphi(z)$ is a function analytic in T, satisfying on L the condition

$$\varphi(x) = w(x,0). \tag{7}$$

Proof. Consider the function

$$v_0(x,y) = \frac{2\beta-1}{y^{2\beta-1}} \int_0^y \frac{w(x,y)\, dy}{y^{2-2\beta}}, \qquad \beta = -\frac{c}{2} + 1 \quad \left(\frac{1}{2} < \beta < 1\right) \tag{8}$$

and prove that $v_0(x,y)$ belongs to the class $N_{2\beta}(T)$.

Indeed, $v_0(x,y)$ is continuous on L, since as $y \to 0$

$$y^{2\beta-1} \to 0, \qquad \int_0^y \frac{w(x,y)\, dy}{y^{2-2\beta}} \to 0, \qquad \lim_{y\to 0} v_0(x,y) = w(x,0).$$

Next, $v_0(x,y)$ satisfies condition (1) for $c = 2\beta$:

$$\frac{\partial^2 v_0}{\partial x^2} + \frac{\partial^2 v_0}{\partial y^2} + \frac{2\beta}{y}\frac{\partial v_0}{\partial y} = (2\beta-1)\left[y^{1-2\beta}\int_0^y y^{2\beta-2} w''_{x^2}\, dy + y^{-1} w'_y\right] =$$

$$= (2\beta-1) y^{1-2\beta} \int_0^y y^{2\beta-2}\left(w''_{x^2} + w''_{y^2} + \frac{2\beta-2}{y} w'_y\right) dy = 0.$$

On the basis of estimate (5) and equation (1), this integral is meaningful. Finally:

$$\lim_{y\to 0} y^{2\beta}\frac{\partial v_0}{\partial y} = (2\beta-1)\lim_{y\to 0}\left[(1-2\beta)\int_0^y y^{2\beta-2}w(x,y)\,dy + y^{2\beta-1}w(x,y)\right] = 0,$$

therefore $v_0(x,y)$, according to paper ([1]), is represented in T in the form (6). Differentiating expression (8), we obtain the required result.

Two solutions $w(x,y)$ and $w^*(x,y)$, defined in T, will be called **conjugate** if they belong to the class $E_c(T)$ and on L satisfy the relations

$$w(x,0) = w^*(x,0), \qquad \lim_{y\to 0} y^c\frac{\partial w}{\partial y} = -\lim_{y\to 0} y^c\frac{\partial w^*}{\partial y}.$$

Theorem 2. In order for a solution $w(x,y) \in E_c^*(T)$ $(-1 < c < 0,\ T \in B)$ to be representable in T in the form

$$w(x,y) = \frac{\gamma(1+c/2)}{c+1}\, y^{-c}\frac{\partial}{\partial y}y^{c+1}\int_0^1 \frac{\varphi[x+iy(1-2\sigma)]\,d\sigma}{[\sigma(1-\sigma)]^{-c/2}} +$$

$$+\gamma\left(1-\frac{c}{2}\right)\left(\frac{y}{1-c}\right)^{1-c}\int_0^1 \frac{\psi[x+iy(1-2\sigma)]\,d\sigma}{[\sigma(1-\sigma)]^{c/2}}, \tag{9}$$

where $\varphi(z)$ and $\psi(z)$ are functions analytic in T, satisfying on L, respectively, relations (7) and (4), it is necessary and sufficient that at least one of the following conditions be fulfilled:

a) the existence in T of a conjugate solution;

b) the existence of a function analytic in T, satisfying condition (7) on the interval L;

c) the existence of a function analytic in T and satisfying condition (4) on the interval L.

Proof. In case b) of the theorem, by means of an analytic function $\varphi(z)$ satisfying condition (7) on L, we construct the first term $w_1(x,y) \in E_c(T)$ of expression (9), satisfying on L the condition $w_1(x,0) = w(x,0)$. Therefore the difference $w(x,y) - w_1(x,y)$, according to Lemma 2, is represented in T in the form of the second term of expression (9), which is what is required for this case.

Case c) is considered analogously.

In case a), considering the sum and the difference of conjugate solutions, we reduce the proof to cases b) and c). The theorem is proved.

In the domain $D = T \cup L \cup \overline{T}$, where $T \in B$, consider the class of solutions $E_c^*(D)$ of equation (1). To it we assign every pair of solutions belonging respectively to the classes $E_c^*(T)$ and $E_c^*(\overline{T})$, forming in D a single function continuous together with the expression $|y|^c \dfrac{\partial w}{\partial y}$.

Theorem 3. Any solution $w(x, y) \in E_c^*(D)$ $(-1 < c < 0)$ is represented in D in the form

$$w(x,y) = \frac{\gamma(1+c/2)}{c+1}|y|^{1-c}\frac{1}{y}\frac{\partial}{\partial y}|y|^{1+c}\int_0^1 \frac{\varphi[x+iy(1-2\sigma)]\,d\sigma}{[\sigma(1-\sigma)]^{-c/2}} +$$

$$+\operatorname{sgn} y \cdot \gamma\Big(1-\frac{c}{2}\Big)\left(\frac{|y|}{1-c}\right)^{1-c}\int_0^1 \frac{\psi[x+iy(1-2\sigma)]\,d\sigma}{[\sigma(1-\sigma)]^{c/2}}, \tag{10}$$

where $\varphi(z)$ and $\psi(z)$ are functions analytic in D, satisfying on L, respectively, condition (7) and the condition

$$\psi(x) = \lim_{y\to 0}\left(\frac{|y|}{1-c}\right)^c \frac{\partial w}{\partial y}. \tag{11}$$

The proof of this theorem completely repeats the proof of the analogous theorem in paper (2).

Remark. All representations considered in the present paper are unique.

On the basis of the preceding, the following is true:

Theorem 4. If for $w(x, y) \in E_c^*(T)$ $(-1 < c < 0,\ T \in B)$ there exists in T a conjugate solution or one of the functions $w(x, 0)$ or $\lim\limits_{y\to 0} |y|^c \dfrac{\partial w}{\partial y}$ is analytically extendable with respect to x to the domain T, then $w(x, y)$ is extended in the form

$$u(z,\zeta) = w\Big(\frac{z+\zeta}{2}, \frac{z-\zeta}{2i}\Big)$$

to the domain

$$\{z\times\zeta \in T\cup L\cup\overline{T}\times T\cup L\cup\overline{T},\quad \operatorname{Im} z > \operatorname{Im}\zeta\}.$$

Moscow Institute of Physics and Technology

Received
27 V 1958

CITED LITERATURE

[1] Yu. P. Krivenkov, DAN, **116**, No. 3 (1957).
[2] Yu. P. Krivenkov, DAN, **116**, No. 4 (1957).
[3] P. Henrici, Proc. Am. Math. Soc., **8**, No. 1 (1957).

Note: Figure translations are in progress. See original paper for figures.