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Abstract

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ON THE SQUARE ROOT OF A LINEAR OPERATOR IN LOCALLY CONVEX SPACES

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1. In a paper by one of the authors ([1]), by studying quadratic forms in the spaces L^q $(1 < q < 2)$, propositions were established on the square root of linear completely continuous operators acting from the space L^q into the space L^p $(p^{-1}+q^{-1}=1)$. These propositions were extended by the other author in ([2]) to a broad class of linear completely continuous operators acting in locally convex spaces. The principal result established in ([2]) can be formulated as follows.

Let the following condition be fulfilled:

(α) E is a real locally convex space; E' is the space strongly conjugate to E ([*]); H is a Hilbert space, dense in E', such that $E \subset H \subset E'$, and the topologies of E and H are compatible, i.e. the topology of the space E majorizes the topology induced in E by the space H, and the topologies of H and E' are also compatible; the bilinear functional $\langle x, y\rangle$, where $y \in E'$ and $x \in E$ or $x \in E''$, coincides with the scalar product in H for $y \in H$.

Then, if E is a semicomplete barrelled space ([3]) and A is a linear completely continuous operator from E' into E, self-adjoint and quasi-positive in H, then the principal square root of A defined in H,

$$A^{1/2}u = \sum_k \sqrt{|\lambda_k|}\,(\varphi_k, u)e_k\varphi_k, \quad e_k = \operatorname{sign}\lambda_k, \tag{1}$$

where φ_k and λ_k are the eigenvectors and the corresponding eigenvalues of the operator A in H, acts completely continuously from H into E and quasi-completely continuously from E' into H.

In this proposition we adhere to the following generally accepted terminology: a linear operator is **completely continuous** if it maps some neighborhood of zero into a relatively bicompact set; a linear operator is **quasi-completely continuous** if it maps every bounded set into a relatively bicompact set.

We note that the formulated proposition in fact also contains the assertion that the operator $A^{1/2}$ is bounded from E' into H, i.e. maps some neighborhood of zero from E' into a bounded set of H. This assertion follows from the following proposition, which we shall also use below.

Lemma. If A is a bounded operator from a locally convex space E into a locally convex space F, then the adjoint operator A' is bounded from F' into E'.

Proof follows from the following two facts: if $A(M) \subset N$, then $A'(N^0) \subset M^0$, where M and N are sets respectively in E and F, and M^0 and N^0 are their polars; if M is a neighborhood of zero in E and N is bounded–

bounded set in F, then M^0 is a bounded set in E' and N^0 is a neighborhood of zero in F'.

In the present paper we establish various propositions on the square root of a linear bounded operator acting from one locally convex space into another, without assuming its complete continuity.

2. Here we shall assume that the spaces E and H satisfy condition (α). This condition, in particular, is fulfilled if $E = L^p(B)$ and $H = L^2(B)$, where B is a set of finite measure in an s-dimensional Euclidean space and $p > 2$. Another such example is $E = D$ and $H = L^2(-\infty, +\infty)$, where D is the space of infinitely differentiable finite functions defined on the line.

Theorem 1. *Let A be a linear bounded operator from E' into E, self-adjoint and positive in H. Then the positive square root $A^{1/2}$ of the operator A, considered in H, is a bounded operator from H into E'' and has a continuous extension $\widetilde{A}^{1/2}$ from E' into H.*

Proof. From the compatibility of the topologies it follows that A is a bounded operator from H into H, and therefore $A^{1/2}$ is a continuous self-adjoint operator in H. Consequently, for $x \in H$ we have $\|A^{1/2}x\|^2 = (A^{1/2}x, A^{1/2}x) = (Ax, x)$. Considering x as an element of E' and taking condition (α) into account, from the last equality we obtain

$$\|A^{1/2}x\|^2 = \langle Ax, x\rangle. \tag{2}$$

Let $\varepsilon > 0$ be given. By the boundedness of A from E' into E there is a neighborhood of zero $U_1 \subset E'$ such that $A(U_1)$ is a bounded set in E; hence the polar $(A(U_1))^0$ is a neighborhood of zero in E', so that for every $x \in U = \varepsilon[(A(U_1))^0 \cap U_1]$ one has $|\langle Ax, x\rangle| \leqslant \varepsilon^2$. Hence, from (2), it follows that for all $x \in U \cap H$ the inequality

$$\|A^{1/2}x\| \leqslant \varepsilon \tag{3}$$

holds.

Inequality (3) shows that the operator $A^{1/2}$, defined on the set H, everywhere dense in the locally convex space E', with values in the complete space H, is continuous; hence, by linearity, it is uniformly continuous on the set H of E'.

Consequently (see (4), Chap. II, § 3, Theorem 1), the operator $A^{1/2}$ has a unique continuous extension $\widetilde{A}^{1/2}$ to all of E', which is linear. This proves the second assertion of the theorem.

Consider the bilinear form $l_x(y) = (A^{1/2}x, y) = (x, A^{1/2}y)$, where $x, y \in H$. By inequality (3) we have $|l_x(y)| \leq \|x\| \, \|A^{1/2}y\| \leq \varepsilon\|x\|$ for every $x \in H$ and arbitrary $y \in H \cap U$. Hence, as above, we conclude that for each $x \in H$ there exists a unique continuous extension $\tilde{l}_x(y)$ to all E' of the functional $l_x(y)$, and therefore $\tilde{l}_x(y) = \langle z_x, y\rangle$, where $z_x \in E''$. By the compatibility of the topologies of the spaces H and E' we have $E'' \subset H$. Since for every $y \in H$ the functionals $\tilde{l}_x(y)$ and $l_x(y)$ coincide, it follows from the preceding and by condition (α) that $(A^{1/2}x, y) = (z_x, y)$, so that $A^{1/2}x = z_x \in E''$, i.e. the operator $A^{1/2}$ acts from H into E''.

In view of the fact that $A^{1/2}x \in E''$ for every $x \in H$ and $\widetilde{A}^{1/2}$ is a continuous operator from E' into H, the functionals $\langle A^{1/2}x, y\rangle$ and $(x, \widetilde{A}^{1/2}y)$ are continuous functionals of y on E' for every $x \in H$. These functionals coincide on the set H, dense in E'; hence, according to the principle of extension of identities (4), it follows that $\langle A^{1/2}x, y\rangle = (x, \widetilde{A}^{1/2}y)$ for all $x \in H$ and $y \in E'$. Taking into account the domains of definition and the ranges of the operators $A^{1/2}$ and $\widetilde{A}^{1/2}$, from the last equality we conclude that $A^{1/2} = (\widetilde{A}^{1/2})'$. Hence, since by what has been proved $\widetilde{A}^{1/2}$ is a bounded operator, according to the lemma it follows that $A^{1/2}$ is a bounded operator from H into E''. The theorem is proved.

3. Here we shall assume that the space E is quasi-barrelled, i.e., every barrel absorbing any bounded set from E is a neighborhood of zero. As is known ([5]), this requirement is necessary and sufficient for the strong topology of the space E'' to induce on E a topology coinciding with the original topology of the space E.

Theorem 2. *Suppose the conditions of Theorem 1 are fulfilled. Then*

$$A = A^{1/2}\widetilde{A}'^{1/2}.$$

Indeed, the operator $C = A^{1/2}\widetilde{A}'^{1/2}$, by Theorem 1, is continuous from E' into E'', while the operator A, continuous from E' into E, in view of the quasi-barrelledness of E, is also continuous from E' into E''. Since the operators A and C coincide on the set H, dense in E', it follows, by the principle of extension of identities, that $A = A^{1/2}\widetilde{A}'^{1/2}$, i.e., A is representable as the product of two operators, of which one—$\widetilde{A}'^{1/2}$—is continuous from E' into H, and the other—$A^{1/2}$—is bounded from the set $\widetilde{A}'^{1/2}(E') \subset H$ into the space E.

Theorem 3. *Suppose the conditions of Theorem 1 are fulfilled. Then the operator A has a continuous extension $\widetilde{A}$ from E''' into E'', representable in the form $\widetilde{A} = A^{1/2}(A^{1/2})'$, where $(A^{1/2})'$ is a continuous operator from E''' into H and $A^{1/2}$ is a bounded operator from H into E''.*

Indeed, since $A^{1/2}$ is a bounded operator from H into E'', by the lemma the operator $(A^{1/2})'$ is continuous from E''' into H. Then from the equality $\langle A^{1/2}x, y\rangle = (x, (A^{1/2})'y)$, where $x \in H,\ y \in E'''$, according to the preceding, in particular when $y \in E'$, we have $(x, \widetilde{A}'^{1/2}y) = \langle A^{1/2}x, y\rangle = (x, (A^{1/2})'y)$ for every $x \in H$. Consequently, $(A^{1/2})'y = \widetilde{A}'^{1/2}y$ for every $y \in E'$, i.e., $(A^{1/2})'$ is a continuous extension to E''' of the operator $\widetilde{A}'^{1/2}$. Hence it follows that $A^{1/2}(A^{1/2})'$ is also a continuous extension of the operator $A^{1/2}\widetilde{A}'^{1/2} = A$.

The theorem just proved generalizes a theorem of V. I. Sobolev ([6]), established by him for Banach spaces.

4. Let a linear bounded operator A from E' into E admit in the space H a representation $A = BB^*$, where B is a linear operator continuous in H, and B^* is its adjoint. Repeating the preceding arguments, we arrive at the following proposition.

Theorem $1'$**.** *Suppose condition (α) is fulfilled. Then B is a bounded operator from H into E, and B^* has a continuous extension $\widetilde{B}^*$ from E' into H.*

We note that, for the operator A considered here, Theorems 2 and 3 remain valid if in their formulations the operator $A^{1/2}$ is everywhere replaced by the operator B, the operator $\widetilde{A}'^{1/2}$ by the operator $\widetilde{B}^*$, and the operator $(A^{1/2})'$ by the operator B', adjoint to the operator B from H into E''.

5. Here we shall assume that condition (α') is fulfilled, which differs from condition (α) in that E is replaced by E' and conversely, so that $E' \subset H \subset E$. In addition, we shall assume that the space E is quasi-barrelled. Repeating then the preceding arguments with some change in the exposition, we arrive at the following propositions.

Theorem 4. *Let A be a linear bounded operator from E into E', self-adjoint and positive in H. Then the positive square root $A^{1/2}$ of the operator A, considered in H, is a bounded operator from H into E' and has a continuous extension $\widetilde{A}^{1/2}$ from E into H.*

Theorem 5. *Suppose the conditions of Theorem 4 are fulfilled. Then the operator A is representable in the form $A = A^{1/2}\widetilde{A}^{1/2}$, where $\widetilde{A}^{1/2}$ is continuous from E into H and $A^{1/2}$ is continuous from H into E'.*

Theorem 6. *Let a linear bounded operator A from E into E' admit in H the representation $A = BB^*$. Then B is a bounded*

operator from H to E', and B^* has a continuous extension $\widetilde{B}^*$ from E to H.

Let us note that Theorem 6 generalizes Theorem 1 of (7), established for Banach spaces.

Let us also note that Theorem 5 remains valid for an operator A from E to E', having in H the representation $A = BB^*$, if in its formulation $A^{1/2}$ is replaced by B and $\widetilde{A}^{1/2}$ by $\widetilde{B}^*$.

6. Suppose condition (α) is satisfied. Consider a linear continuous operator A from E' to E, self-adjoint in H. Denote by A_+ and A_- the positive and negative parts of A in H, and suppose that A_+ and $|A_-|$ have extensions $\widetilde{A}_+$ and $|\widetilde{A}_-|$, bounded from E' to E. Then, using Theorems 1 and 2, one can prove that the operator A has the representation

$$A = A^{1/2}|\widetilde{A}|^{1/2},$$

where $|\widetilde{A}|^{1/2}$ is the extension of the square root of

$$|A| = A_+ + |A_-|,$$

continuous from E' to H, and

$$A^{1/2} = A_+^{1/2} - |A_-|^{1/2}$$

is the principal square root of A in H, bounded from $|\widetilde{A}|^{1/2}(E') \subset H$ to E. There is also an analogue of Theorem 3; i.e., the operator A has a continuous extension $\widetilde{A}$ from E''' to E'', representable in the form

$$\widetilde{A} = A^{1/2}(|A|^{1/2})',$$

where $(|A|^{1/2})'$ is continuous from E''' to H and $A^{1/2}$ is bounded from H to E''. If condition (α') is satisfied, then propositions analogous to those of item 5 hold.

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Note: Figure translations are in progress. See original paper for figures.