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1958

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Abstract

Full Text

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Estimates in L_p of Solutions of Elliptic Systems

(Presented by Academician V. I. Smirnov on 4 VII 1958)

1. In the present note a generalization is given, in one direction, of the results set forth in the notes ($^{1-3}$) and in the article of Gagliardo (^{4}). In doing so we adhere to the terminology adopted in (^{2}, ^{3}).
2. **Theorem 1.** Let l be a natural number, $1 < p < +\infty$, and let Ω be a bounded domain of the n-dimensional space E_n, bounded by a surface S that is continuously differentiable a finite number $l+1$ of times. Let $v = v(x) \in W_p^{(l)}(\Omega)$.

Then, for $k = 0, 1, \dots, l-1$, the normal derivatives $\partial^k v / \partial \nu^k$, as functions of a point of the surface S, belong to the spaces $W_p^{(l-k-1/p)}(S)$. Moreover,

$$\left\| \frac{\partial^k v}{\partial \nu^k} \right\|_{W_p^{(l-k-1/p)}(S)} \leq C_1 \|v\|_{W_p^{(l)}(\Omega)}, \tag{1}$$

where C_1 depends only on Ω.

Conversely, if functions $\varphi_k(x') \in W_p^{(l-k-1/p)}(S)$ $(k = 0, 1, \dots, l-1)$ are given, then there exists a function $\bar{v} \in W_p^{(l)}(\Omega)$ satisfying the boundary conditions

$$\left. \frac{\partial^k \bar{v}}{\partial \nu^k} \right|_S = \varphi_k \quad (k = 0, 1, \dots, l-1). \tag{2}$$

Moreover,

$$\|\bar{v}\|_{W_p^{(l)}(\Omega)} \leq C_2 \sum_{k=0}^{l-1} \|\varphi_k\|_{W_p^{(l-k-1/p)}(S)}, \tag{3}$$

where C_2 depends only on Ω.

For $p = 2$, a more general result (obtained with the aid of the Fourier transform and Parseval' s equality) is given in note (^{2}). For $l = 1$, Theorem 1 was proved by Gagliardo (^{4}).

The first part of our theorem follows directly from Gagliardo' s theorem. We shall therefore dwell in more detail on the proof of its second part.

It suffices to assume that Ω is the half-space $x_n > 0$. Assuming that $\psi_s(x') \in W_p^{(l-s-1/p)}(E_{n-1})$ $(0 \le s \le l-1)$, consider, for $x_n > 0$, the function

$$v_s(x) = K_n e^{-x_n} \frac{x_n^{s+1}}{s!} \int_{E_{n-1}} \frac{\psi_s(x'+t')\,dt'}{(|t'|^2 + x_n^2)^{n/2}},$$

where

$$K_n = \left(\int_{E_{n-1}} \frac{dt'}{(|t'|^2+1)^{n/2}} \right)^{-1}.$$

It is easy to see that

$$\left.\frac{\partial^k v_s}{\partial x_n^k}\right|_{x_n=0} = 0 \quad (0 \le k \le s-1); \qquad \left.\frac{\partial^s v_s}{\partial x_n^s}\right|_{x_n=0} = \psi_s(x').$$

Let

$$\frac{n-2}{p} + 1 < \varepsilon < \frac{n-1}{p} + 1.$$

Applying Hölder' s inequality, we obtain:

$$\begin{aligned} |v_s(x)| &\le C_1 e^{-x_n} x_n^{s-n+1} \int_{E_{n-1}} \frac{|\psi_s(x'+t')|}{(1+|t'|^2/x_n^2)^{n/2}}\,dt' \\ &\le C_1 e^{-x_n} x_n^{s-n+1} \left[\int_{E_{n-1}} \frac{|\psi_s(x'+t')|^p}{(1+|t'|^2/x_n^2)^{\varepsilon p/2}}\,dt' \right]^{1/p} \times \\ &\times \left[\int_{E_{n-1}} \frac{dt'}{(1+|t'|^2/x_n^2)^{\frac{n-\varepsilon}{2}p'}} \right]^{1/p'} \\ &= C_2 e^{-x_n} x_n^{s-\frac{n-1}{p}} \left[\int_{E_{n-1}} \frac{|\psi_s(x'+t')|^p\,dt'}{(1+|t'|^2/x_n^2)^{\varepsilon p/2}} \right]^{1/p}. \end{aligned}$$

Hence

$$\begin{aligned}\int_0^{+\infty} dx_n \int_{E_{n-1}} |v_s|^p\,dx' \le C_3 \int_0^{+\infty} e^{-px_n} x_n^{ps-n+1}\,dx_n \int_{E_{n-1}} \frac{dt'}{(1+|t'|^2/x_n^2)^{\varepsilon p/2}} \times \\ \times \int_{E_{n-1}} |\psi_s(x'+t')|^p\,dx' \\ = C_3 \int_0^{+\infty} e^{-px_n} x_n^{ps}\,dx_n \times \\ \times \int_{E_{n-1}} \frac{dt'}{(|t'|^2+1)^{\varepsilon p/2}} \int_{E_{n-1}} |\psi_s(x')|^p\,dx' \\ = C_4 \int_{E_{n-1}} |\psi_s(x')|^p\,dx'.\end{aligned} \tag{4}$$

Next we have

$$\begin{aligned}D^l_{x'} v_s &= K_n e^{-x_n} \frac{x_n^{s+1}}{s!} \int_{E_{n-1}} D^{l-s-1}\psi_s(x'+t')\, D^{s+1}_{t'} \left[\frac{1}{(|t'|^2+x_n^2)^{n/2}}\right] dt' \\ &= K_n e^{-x_n} \frac{x_n^{s+1}}{s!} \int_{E_{n-1}} [D^{l-s-1}\psi_s(x'+t') - D^{l-s-1}\psi_s(x')]\, D^{s+1}_{t'} \left(\frac{1}{(|t'|^2+x_n^2)^{n/2}}\right) dt'.\end{aligned}$$

Hence, with the aid of Hölder' s inequality, we obtain

$$\begin{aligned}\int_0^{+\infty} dx_n \int_{E_{n-1}} |D^l_{x'} v_s|^p\,dx' &\le C_1 \int_0^{+\infty} dx_n \int_{E_{n-1}} \left[\int_{E_{n-1}} \frac{|D^{l-s-1}\psi_s(x'+t') - D^{l-s-1}\psi_s(x')|}{(|t'|^2+x_n^2)^{n/2}}\, dt'\right]^p dx' \\ &\le C_2 \int_{E_{n-1}} dx' \int_0^{+\infty} x_n^{-(n-1)-p}\,dx_n \int_{E_{n-1}} \frac{|D^{l-s-1}\psi_s(x'+t') - D^{l-s-1}\psi_s(x')|^p}{(1+|t'|^2/x_n^2)^{\varepsilon p/2}}\,dt' \\ &\le C_3 \int_{E_{n-1}} dx' \int_{E_{n-1}} \frac{|D^{l-s-1}\psi_s(x'+t') - D^{l-s-1}\psi_s(x')|^p}{|t'|^{n-2+p}}\,dt'.\end{aligned} \tag{5}$$

In a somewhat more complicated way one obtains (5) for $D^l_{x_n} v_s$. From (4) and (5) it follows that

$$\|v_s\|_{W_p^{(l)}(x_n>0)} \le C_5 \|\psi_s\|_{W_p^{(l-s-1/p)}(E_{n-1})}.$$

By the method described above we construct the function $v_0(x)$ from $\psi_0 = \varphi_0$, and v_s from

$$\psi_s = \varphi_s - \sum_{k=0}^{s-1} \frac{\partial^s v_k}{\partial x_n^s}\bigg|_{x_n=0} \qquad (s = 1, 2, \dots, l-1).$$

Then, from the first part of Theorem 1 and the estimates obtained for v_s, it is easy to obtain that the function

$$\bar{v} = \sum_{s=0}^{l-1} v_s$$

satisfies conditions (2) and (3).

3. From Theorem 1 of the present work and Theorem 3 of note (3) there follows the following proposition.

Theorem 2. *Let $L = L(x, \partial/\partial x)$ be an elliptic differential operator of order $2k$, defined in $\overline{\Omega} = \Omega + S$; let $R_\mu = R_\mu(x', \partial/\partial x)$ be differential operators of orders m_μ $(\mu = 1, 2, \dots, k)$, defined on S and connected with L by condition (L) (see (3)). Let l be a natural number $\geq 2k$ and $m_\mu \leq l-1$ $(\mu = 1, 2, \dots, k)$.*

If the coefficients of the differential operators L and R_μ have $l-2k$ bounded derivatives and their leading coefficients are continuous in the domains of their definition, and if the surface is $l+1$ times continuously differentiable, then for any function $u = u(x) \in W_p^{(l)}(\Omega)$ the inequality

$$C_1 \left[\|Lu\|_{W_p^{(l-2k)}(\Omega)} + \sum_{\mu=1}^{k} \|R_\mu u\|_{W_p^{(l-m_\mu-1/p)}(S)} \right] \leq$$

$$\leq \|u\|_{W_p^{(l)}(\Omega)} \leq C_2 \left[\|Lu\|_{W_p^{(l-2p)}(\Omega)} + \sum_{\mu=1}^{k} \|R_\mu u\|_{W_p^{(l-m_\mu-1/p)}(S)} + \|u\|_{L_p(\Omega)} \right], \quad (6)$$

holds, where C_1 and C_2 are positive constants independent of $u(x)$.

4. Both theorems admit a generalization to the case when Ω is an unbounded domain and S is an unbounded sufficiently smooth surface without boundary.

Let $\overline{\Omega}$ be a closed, but, generally speaking, unbounded domain of the space E_n. A function $f(x)$, defined in $\overline{\Omega}$, will be called **continuous in this domain** if it is continuous at each of its points and if there exists a finite limit $f(x)$ as $|x| \to +\infty$.

Let σ be a finite or infinite $(n-1)$-dimensional surface given by the equation $x = x(\gamma')$ $(\gamma' = (\gamma_1, \dots, \gamma_{n-1}))$, where the vector function $x(\gamma')$ is defined in some domain $d(\sigma)$ of the $(n-1)$-dimensional Euclidean space of the points γ'.

Denote by $\nu = \nu(\gamma')$ the unit vector of the normal to σ. We shall say that $\sigma \in K^{(l)}$ if, for some $b > 0$, the domain $\Delta(\sigma)$, defined by the relations $\gamma' \in d(\sigma)$, $|\gamma_n| < b$, is mapped one-to-one onto some n-dimensional neighborhood $D(\sigma)$ of the surface σ by means of

$$x = x(\gamma') + \nu(\gamma')\gamma_n = x(\gamma), \qquad \gamma = (\gamma', \gamma_n),$$

and, moreover, so that $x = x(\gamma)$ has in $\Delta(\sigma)$, and $\gamma = \gamma(x)$ has in $D(\sigma)$, bounded derivatives up to order l, while these vector functions themselves are continuous in the above-indicated sense.

Let $\delta > 0$. Denote by Ω_δ the subdomain of the domain Ω consisting of those points of the domain Ω whose distances to its boundary are greater than δ.

We shall say that an $(n-1)$-dimensional surface without boundary S of the space E_n is a **surface of class** $R^{(l)}$ if it can be covered by a finite number of surfaces $\sigma_i^{(0)}$ $(i = 1, 2, \dots, q)$ of class $K^{(l)}$ with the following properties: a) each point of S belongs to at least one of the $\sigma_i^{(0)}$; b) each point of any one of the $\sigma_i^{(0)}$ belongs to S; c) for some

for $\delta > 0$, the parts $\sigma_i^{(1)}$ of the surfaces $\sigma_i^{(0)}$ lying in $D_\delta(\sigma_i^{(0)})$ also possess properties a) and b).

The definition of the spaces $W_p^{(l)}$ by means of parametrization is easily extended to surfaces $S \in R^{(l)}$.

We can now say that Theorem 1 is valid for the case where Ω is an infinite domain bounded by a surface of class $R^{(l)}$. In the same case Theorem 2 will also be valid if it is further added that the continuity of the leading coefficients of the operators L and R_μ is understood in the sense indicated above.

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Received
2 VII 1958

CITED LITERATURE

1. L. N. Slobodetskii, V. M. Babich, DAN, 106, No. 4 (1956).
2. L. N. Slobodetskii, DAN, 118, No. 2 (1958).
3. L. N. Slobodetskii, DAN, 120, No. 2 (1958).
4. E. Gagliardo, Rend. Sem. Mat. di Padova, 27 (1957).