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Abstract

Full Text

Mathematics

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On the Behavior of Cycles Not Homologous to Zero under a Mapping of an n -Dimensional Manifold into n -Dimensional Euclidean Space

(Presented by Academician P. S. Aleksandrov, 27 VI 1957)

§ 1. The answer to many questions concerning the behavior of cycles under a continuous mapping of an n -dimensional closed orientable manifold M_1^n into another orientable manifold M_2^n with nonzero degree is given by Hopf's work⁽¹⁾ and by K. A. Sitnikov's theorem on inverse isomorphisms⁽²⁾, which, in particular, strengthens part of Hopf's work.

P. S. Aleksandrov posed the following problem: to prove that under a mapping of an n -dimensional closed orientable manifold M^n into n -dimensional Euclidean space or into an n -dimensional orientable manifold with degree zero, there exist "collapsing" cycles (i.e., cycles mapped to zero) of all dimensions. The existence of such cycles homologous to zero in M_1^n in all dimensions was proved by me in⁽³⁾.

Hopf proved⁽¹⁾ that, for a mapping of M_1^n into M_2^n of degree zero (M_1^n and M_2^n are closed orientable manifolds), the following estimate holds:

$$p_f^s + p_f^{n-s} \leq \frac{p^s(M_1^n) + p^{n-s}(M_1^n)}{2},$$

where $p^s(M_1^n)$ is the rank of the Betti group of dimension s of the manifold M_1^n ; p_f^s is the maximal number of homologically independent cycles in M_2^n of dimension s that are images of cycles from M^n under the mapping f .

In connection with this result, K. A. Sitnikov raised the question of the number of cycles not homologous to zero in M^n that are mapped to zero (the precise definition will be given below). The present note is devoted to this question.

The field of rational numbers is taken as the coefficient group. We shall use the following notation: M^n (sometimes with a subscript below) is an n -dimensional closed orientable manifold; R^n is n -dimensional Euclidean space; $\Delta^s(K)$ is the s -dimensional Betti group of the complex K ; $p^s(K)$ is the rank of the group $\Delta^s(K)$. A polyhedron and a complex that is its triangulation will usually be denoted by the same letter.

In § 2 a theorem is given that strengthens the above-mentioned result of Hopf; in § 3 the case of a mapping of M^n into R^n or into a part of M_1^n is considered.

§ 2. **Definition.** Let f be a continuous mapping of M^n into R^n or into M_1^n , and let ξ^s be a nonzero s -dimensional homology class of M^n . We shall say that **there exist cycles of this homology class,**

mapped by means of f to zero, if in R^n (in M_1^n) there exists a polyhedron L of dimension $\leq s$ such that, for every $\varepsilon > 0$, the set $f^{-1}(O(L, \varepsilon))$ is the carrier of a cycle from ξ^s , and the image of this cycle is homologous to zero in $O(L, \varepsilon)$.

Let the maximum number of independent s -dimensional homology classes such that there exist cycles from these homology classes which are mapped by f to zero be denoted by r_f^s . By q_f^s denote the maximum number of independent s -dimensional cycles of the closed set $f(M^n)$ that are images of cycles from M^n under the mapping f , i.e. q_f^s is the rank of the subgroup $\Delta^s(f(M^n))$ onto which the group $\Delta^s(M^n)$ is mapped. It is clear that p_f^s —the rank of the subgroup $\Delta^s(M_1^n)$ onto which $\Delta^s(M^n)$ is mapped—does not exceed q_f^s , and for a mapping into R^n , $p_f^s \equiv 0$. It is easy to construct a mapping of an orientable surface of genus p into R^2 for which $q_f^1 = p$.

Theorem 1. Let f be a continuous mapping of M^n into R^n or into M_1^n of degree zero, and let $p^s(M^n) \neq 0$. Then

$$q_f^s + q_f^{n-s} \leq \frac{p^s(M^n) + p^{n-s}(M^n)}{2},$$

$$q_f^s \leq r_f^{n-s}, \quad q_f^{n-s} \leq r_f^s.$$

Plan of proof of Theorem 1 (we consider $p^s(M^n) = 1$ and a mapping into R^n). Take $R^m \supset R^n$, $m > 2n + 1$, and let f_k be a sequence of simplicial topological mappings of M^n into R^m approximating f . Let ζ^s and ζ^{n-s} be generators of the groups $\Delta^s(M^n)$ and $\Delta^{n-s}(M^n)$, and let z^s be a cycle from ζ^s . If $f(z^s) \sim 0$ on $f(M^n)$, then take in $R^m \setminus f(M^n)$ a cycle z^{m-s-1} linked with $f(z^s)$, and let x^{m-s} be a chain bounding z^{m-s-1} . Denote by L^{n-s} the polyhedron of dimension $n - s$ which is the intersection of the body x^{m-s} with R^n . The chain x^{m-s} cuts out on the manifold $f_k(M^n)$ a cycle $z_k^{n-s} = f_k(z_{1,k}^{n-s})$, where $z_{1,k}^{n-s} \in \zeta^{n-s}$. The cycle z_k^{n-s} lies in $O(L^{n-s}, \varepsilon_k)$ and is homologous to zero in this neighborhood, with $\varepsilon_k \rightarrow 0$ as $k \rightarrow \infty$.

It is easy to construct a mapping of some manifold M_1^n into M_2^n of degree zero for which $p_f^s + p_f^{n-s} = p^s(M_1^n)$, i.e. the number of homologically independent cycles of dimensions s and $n - s$ in M_1^n whose images are not homologous to zero on $f(M_1^n)$ is equal to $p^s(M_1^n)$. Therefore there exist mappings of M_1^n into R^n or into M_2^n of degree zero for which the total number of homologically independent cycles of dimensions s and $n - s$ mapped to zero (under any definition of a “collapsing” cycle) does not exceed $p^s(M_1^n)$.

§ 3. **Mapping of M^n into R^n or into a part M_1^n .** In the study of sets containing cycles not homologous to zero on M^n , the following simple but important lemma, communicated to me by K. A. Sitnikov, is fundamental.

Lemma. Let $p^s(M^n) \neq 0$. If A is a closed subset of the manifold M^n that splits a nonzero homology class ζ^s (in the sense that $M^n \setminus A$ contains no carriers of cycles from ζ^s), then A contains an $(n-s)$ -dimensional cycle z^{n-s} having nonzero intersection index with cycles from ζ^s .

Definition of the numbers ω_f^s and $\bar{\mu}_f^s$. Let f be a continuous mapping of M^n into a polyhedron K . Denote by μ_f^s the maximum number of such independent non-

dependent elements of the group $\Delta^s(M^n)$, such that there exist carriers of cycles from these homology classes which are mapped by means of f into polyhedra containing no nonzero s -dimensional cycles.

By ψ_f^s we denote the maximal number of such independent elements of the group $\Delta^s(M^n)$ such that, for every $\varepsilon > 0$, there exist carriers of cycles from these homology classes whose images, by means of an ε -shift, are carried into a polyhedron containing no nonzero s -dimensional cycles.

Theorem 2. Let f be a continuous mapping of M^n into R^n or into a part M_1^n of it, and let $p^s(M^n) \neq 0$. Then

$$\psi_f^s + \bar{\psi}_f^{n-s} \geq p^s(M^n).$$

We outline the plan of the proof of Theorem 2 for a mapping into R^n . Introduce the following notation: by l denote the axis x_1 , by R^{n-1} the $(n-1)$ -dimensional plane $x_1 = 0$; by $I_1, I_2, \dots, I_k, \dots$ a sequence of refining cubical decompositions of R^{n-1} , where the edge length of a cube of I_k is $a/2^k$; by C_k^{s-1} the $(s-1)$ -dimensional skeleton of I_k , $C_k^{s-1} \subset C_{k+1}^{s-1}$; by L_k^s the infinite polyhedron $C_k^{s-1} \times l$, where L_k^s contains no nonzero s -dimensional cycles and $L_k^s \subset L_{k+1}^s$; by L_k^{n-s} the infinite polyhedron $C_{1,k}^{n-s-1} \times l$, where $C_{1,k}^{n-s-1}$ is the $(n-s-1)$ -dimensional skeleton of the cubical decomposition dual to I_k ; L_k^{n-s} contains no $(n-s)$ -dimensional cycles. If a compact set $\Phi \subset R^n \setminus L_k^s$, then Φ , by means of the operation of "sweeping out," is carried into L_k^{n-s} , the magnitude of this shift being

$$\leq \frac{a\sqrt{n-1}}{2^k}.$$

Denote by r_k the maximal number of independent $(n-s)$ -dimensional homology classes such that there exist carriers of cycles from these classes whose images lie in $R^n \setminus L_k^s$. It is clear that $r_k \geq r_{k+1}$. Let $r = \inf_k r_k$; there exists a k_0 such that for $k \geq k_0$, $r_k = r$. It is proved that $\bar{\psi}_f^{n-s} \geq r$. The set $f^{-1}(L_k^s)$, $k \geq k_0$, splits, for any choice of a basis of the group $\Delta^{n-s}(M^n)$, no fewer than

$(p^{n-s}(M^n) - r)$ elements of this basis. Hence, relying on the lemma, we prove that $\psi_f^s \geq p^{n-s}(M^n) - r$. Combining this with the inequality $\overline{\psi}_f^{n-s} \geq r$ gives the desired result.

Theorem 3. *Let f be a continuous mapping of M^n into R^n or into a part M_1^n of it, and let $p^1(M^n) \neq 0$. Then*

$$\psi'_f + \psi_f^{n-1} \geq p^1(M^n).$$

In conclusion I express my sincere gratitude to P. S. Aleksandrov and K. A. Sitnikov for the valuable advice they gave me while I was working on this question.

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