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Abstract

Full Text

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ON THE IMPOSSIBILITY OF CONSTRUCTING A LINEAR POLYNOMIAL OPERATOR GIVING AN APPROXIMATION OF THE ORDER OF THE BEST APPROXIMATION

(Presented by Academician A. N. Kolmogorov on 13 II 1958)

Mathematics

1°. For definiteness we shall first consider the space $\widetilde{C}$ of all continuous 2π-periodic functions $f(x)$ with norm

$$\|f\|_{\widetilde{C}} = \max_{0 \leq x < 2\pi} |f(x)|.$$

Put $f_t(x) = f(x+t)$.

It is known that for every $f \in \widetilde{C}$ there exists a unique polynomial of best approximation of order n, which we shall denote by $r_n(f,x)$. The polynomial $r_n(f,x)$ may be regarded as an operator that assigns to each $f \in \widetilde{C}$ its polynomial of best approximation of order n. The operator $r_n(f,x)$ is nonlinear.

At present there is as yet no method for computing polynomials of best approximation; therefore the following question is natural:

Does there exist a **linear operator** $U_n(f,x)$ possessing the properties:

1) $U_n(f,x)$ maps $\widetilde{C}$ into $\widetilde{C}$;
2) for every $f \in \widetilde{C}$, $U_n(f,x)$ is a polynomial of order $\leq n$;
3) for every $f \in \widetilde{C}$,

$$\|f(x) - U_n(f,x)\| = O(E_n),$$

where $E_n(f) = E_n = \|f(x) - r_n(f,x)\|$?

In the present note it is proved that this question must be answered in the negative.

2°. **Theorem 1.** *There does not exist a linear operation $U_n(f,x)$ satisfying conditions 1)—3).*

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Proof. Suppose that there exists an operation $U_n(f,x)$ satisfying conditions 1)–3). Then from condition 3), taking into account that $E_n(f)=0$ when f is a trigonometric polynomial of order not exceeding n, it follows that

$$U_n(f,x)\equiv f(x), \tag{1}$$

when f is a trigonometric polynomial of order not exceeding n.

But it is known ([1]) that for every linear operation satisfying conditions 1)–2) and equality (1), the equality

$$\frac{1}{2\pi}\int_0^{2\pi} U_n(f_t, x-t)\,dt = S_n(f,x), \tag{2}$$

holds, where $S_n(f,x)$ is the partial sum of order n of the Fourier series of the function $f(x)$.

It is easy to see that

$$\frac{1}{2\pi}\int_0^{2\pi} (f_t,\ x-t)\,dt = f(x), \tag{3}$$

where (f,x) is the value of the function f at the point x.

From (2) and (3) it follows that

$$\frac{1}{2\pi}\int_0^{2\pi} (U_n(f_t)-f_t,\ x-t)\,dt = S_n(f,x)-f(x). \tag{4}$$

Since the operator $U_n(f,x)$ satisfies condition 3), there exists a constant C such that

$$\|U_n(f_t)-f_t\| \le CE_n(f_t), \qquad -\infty<t<\infty. \tag{5}$$

We note that

$$E_n(f_t)=E_n(f), \qquad -\infty<t<\infty.$$

Therefore inequality (5) takes the form

$$\|U_n(f_t,x)-f_t(x)\| \le CE_n(f), \qquad -\infty<t<\infty,$$

and then from equality (4) it follows that

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$$|S_n(f,x) - f(x)| \leq CE_n(f), \qquad -\infty < x < \infty. \tag{6}$$

Inequality (6) contradicts the classical result of du Bois-Reymond from the theory of Fourier series. Theorem 1 is proved.

An analogous theorem can be proved in the case of the space C of all functions $f(x)$ continuous on the segment $[-1,1]$, with norm $\|f\| = \max_{-1\leq x\leq 1} |f(x)|$.

3°. In ([2]) it is proved that formula (2) is valid for functional spaces of type E, a particular case of which is the space $\widetilde{C}$. Therefore Theorem 1 admits the following generalization:

Theorem 2. *Let a functional space of type E be such that, for at least one function $f \in E$, the relation*

$$\overline{\lim_{n\to\infty}} \|f - S_n(f)\| = \infty$$

holds. Then there does not exist a linear operation $U_n(f,x)$ satisfying the conditions:

1) *$U_n(f,x)$ maps E into E;*

2) *for every $f \in E$, $U_n(f,x)$ is a polynomial of order $\leq n$;*

3) *for every $f \in E$,*

$$\|U_n(f) - f\|_E = O(E_n(f)),$$

where

$$E_n(f) = \inf_{T_n \in \Pi_n} \|f - T_n\|_E$$

and T_n ranges over the set Π_n of all trigonometric polynomials of order $\leq n$.

The space $\widetilde{L}$ of all summable 2π-periodic functions with norm

$$\|f\| = \frac{1}{2\pi} \int_0^{2\pi} |f(t)|\, dt$$

is a functional space of type E. Moreover, it is known that there exist functions $f \in \widetilde{L}$ satisfying the condition

$$\overline{\lim_{n\to\infty}} \int_0^{2\pi} |f - S_n(f)|\, dx = \infty.$$

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Therefore, from Theorem 2 it follows:

Theorem 3. There does not exist a linear operation satisfying the conditions:

1) $U_n(f,x)$ maps $\widetilde{L}$ into $\widetilde{L}$;

2) for every $f \in \widetilde{L}$, $U_n(f,x)$ is a trigonometric polynomial of order $\le n$;

3) for every $f \in \widetilde{L}$,

$$\int_0^{2\pi} |f(x) - U_n(f,x)|\, dx = O(E_n(f)),$$

where

$$E_n = E_n(f) = \inf_{T_n \in \Pi_n} \frac{1}{2\pi} \int_0^{2\pi} |f(x) - T_n(x)|\, dx.$$

Remark. It is known ([3]) that formula (2) generalizes to the case of functions of many variables. In view of this, Theorem 2 can also be extended to the case of functions of many variables.

4°. It is well known that, for any fixed p satisfying the inequalities $0 < p < 1$, one can construct a linear polynomial operator having properties 1) and 2) and satisfying, for every $f \in \widetilde{C}$, the condition

$$\|f(x) - U_n(f,x)\| = O(E_{[pn]}(f)),$$

where $[pn]$ is the integer part of the number pn ([4,5]).

Therefore, in connection with Theorem 1 the following problem arises.

Let an arbitrary sequence of positive numbers $\{p_n\}_{n=1}^{\infty}$ be given, satisfying the inequalities $0 < p_n \le 1$, $n = 1, 2, ...$, and let $\lim_{n\to\infty} p_n = 1$. The question is whether one can construct a linear polynomial operator $U_n(f,x)$ having properties 1) and 2) and satisfying, for every $f \in \widetilde{C}$, the condition

$$3') \quad \|f(x) - U_n(f,x)\| = O(E_{[p_n n]}(f)).$$

The solution of this problem is given by the theorem:

Theorem 4. There does not exist a linear operation $U_n(f,x)$ satisfying conditions 1)–2) and 3′).

In connection with Theorem 3 the following theorem is appropriate:

Theorem 5. For any fixed $f \in \widetilde{C}$ one can construct such a linear polynomial operator $U_n(f,x)$ of order n that

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$$|f(x) - U_n(f,x)| \leq E_n(f), \qquad n = 0, 1, 2, \dots, \quad -\infty < x < \infty. \tag{7}$$

This theorem follows directly from Marcinkiewicz' s theorem ([6]), according to which, for any $f \in \widetilde{C}$, one can find such a matrix of interpolation nodes that the corresponding Lagrange interpolation process satisfies inequality (7).

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CITED LITERATURE

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[6] I. P. Natanson, *Constructive Theory of Functions*, 1949.

Note: Figure translations are in progress. See original paper for figures.

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