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Abstract

Full Text

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A SUFFICIENT CONDITION FOR THE CONVERGENCE OF THE METHOD OF AVERAGING FUNCTIONAL CORRECTIONS

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MATHEMATICS

Consider the linear Fredholm integral equation of the second kind

$$y(x) = \varphi(x) + \lambda \int_a^b K(x,\xi) y(\xi)\, d\xi \qquad (0 < |\lambda| < \infty). \tag{1}$$

Assume that the functions $\varphi(x)$ and $K(x,\xi)$ are real and belong to the space $L^2(a,b)$. Let equation (1) have a unique solution for a certain value of the parameter λ.

The method of averaging functional corrections, set forth in ($^{1-4}$), consists in the following: in the first approximation we put

$$y_1(x) = \varphi(x) + \lambda\alpha_1 \int_a^b K(x,\xi)\, d\xi, \tag{2}$$

where

$$\alpha_1 = \frac{1}{h}\int_a^b y_1(x)\, dx \qquad (h = b - a > 0). \tag{3}$$

From equalities (2) and (3) we determine α_1:

$$\alpha_1 = \frac{1}{D(\lambda)} \int_a^b \varphi(x)\, dx, \tag{4}$$

where

$$D(\lambda) = h - \lambda \int_a^b \int_a^b K(x,\xi)\, d\xi\, dx.$$

In the n-th approximation we put

$$y_n(x) = \varphi(x) + \lambda \int_a^b K(x,\xi)(y_{n-1}(\xi) + \alpha_n)\, d\xi, \tag{5}$$

where

$$\alpha_n = \frac{1}{h} \int_a^b \delta_n(x)\, dx; \tag{6}$$

$$\delta_n(x) = y_n(x) - y_{n-1}(x) \qquad (n = 2, 3, \dots). \tag{7}$$

From equalities (5), (6), and (7) we obtain

$$\delta_n(x) = \lambda \int_a^b K(x,\xi)(\delta_{n-1}(\xi) - \alpha_{n-1})\, d\xi + \lambda\alpha_n \int_a^b K(x,\xi)\, d\xi; \tag{8}$$

$$\alpha_n = \frac{\lambda}{D(\lambda)} \int_a^b \int_a^b K(x,\xi)(\delta_{n-1}(\xi) - \alpha_{n-1})\, d\xi\, dx \qquad (n = 2, 3, \dots). \tag{9}$$

Let $D(\lambda) \neq 0$. Then from the assumptions concerning the functions $\varphi(x)$ and $K(x,\xi)$ it follows that all the functions $y_n(x)$, and hence also all the functions $\delta_n(x)$, belong to the space $L^2(a,b)$.

The functions $\delta_n(x)$ and α_n can be represented in the form

$$\delta_n(x) = \lambda \int_a^b (K(x,\xi) - M(x))(\delta_{n-1}(\xi) - \lambda h \alpha_{n-1} t)\, d\xi + \lambda \alpha_n h M(x); \tag{8'}$$

$$\alpha_n = \frac{\lambda}{D(\lambda)} \int_a^b \int_a^b (K(x,\xi) - M(x))(\delta_{n-1}(\xi) - \lambda h \alpha_{n-1} t)\, d\xi\, dx, \tag{9'}$$

where

$$M(x) = \frac{1}{h} \int_a^b K(x,\xi)\, d\xi,$$

t is an arbitrary parameter.

Subtracting $\lambda h \alpha_n t$ from both sides of equality $(8')$, squaring the result obtained, and integrating with respect to x, we then obtain

$$\Phi_n(t) = \int_a^b (\delta_n(x) - \lambda h \alpha_n t)^2\, dx =$$

$$= \lambda^2 \int_a^b \left\{ \int_a^b (K(x,\xi) - M(x))(\delta_{n-1}(\xi) - \lambda h \alpha_{n-1} t)\, d\xi + \alpha_n h (M(x) - t) \right\}^2 dx.$$

Using Minkowski' s inequality, we obtain:

$$\{\Phi_n(t)\}^{1/2} \leq |\lambda| \left\{ \int_a^b \left[\int_a^b (K(x,\xi) - M(x))(\delta_{n-1}(\xi) - \lambda h \alpha_{n-1} t)\, d\xi \right]^2 dx \right\}^{1/2} +$$

$$+ |\lambda| h \left\{ \alpha_n^2 \int_a^b (M(x) - t)^2\, dx \right\}^{1/2}.$$

Applying the Cauchy-Bunyakovsky inequality, we finally obtain the inequality

$$\{\Phi_n(t)\}^{1/2} \leq |\lambda| \left\{ \int_a^b \int_a^b (K(x,\xi) - M(x))^2\, d\xi\, dx \right\}^{1/2} \cdot \{\Phi_{n-1}(t)\}^{1/2} +$$

$$+ |\lambda| h \left\{ \alpha_n^2 \int_a^b (M(x) - t)^2\, dx \right\}^{1/2}. \tag{10}$$

From $(9')$ we have:

$$\alpha_n^2 \leq \frac{\lambda^2 h}{D^2(\lambda)}\, \Phi_{n-1}(t) \int_a^b \int_a^b (K(x,\xi) - M(x))^2\, d\xi\, dx. \tag{11}$$

On the basis of (11), from (10) we obtain the relation

$$\{\Phi_n(t)\}^{1/2} \leq |\lambda| \{\Phi_{n-1}(t)\}^{1/2} \left\{ \int_a^b \int_a^b (K(x,\xi) - M(x))^2\, d\xi\, dx \right\}^{1/2} \times$$

$$\times \left\{ 1 + \frac{h^{3/2}|\lambda|}{|D(\lambda)|} \left[\int_a^b (M(x) - t)^2\, dx \right]^{1/2} \right\}.$$

Put

$$t = K = \frac{1}{h^2} \int_a^b \int_a^b K(x, \xi)\, d\xi\, dx;$$

then

$$\Phi_n(K) \leq \mathcal{L}^2 \Phi_{n-1}(K), \tag{12}$$

where

$$\mathcal{L}^2 = \lambda^2 \int_a^b \int_a^b (K(x,\xi) - M(x))^2\, d\xi\, dx \left\{ 1 + \frac{h^{3/2}|\lambda|}{|D(\lambda)|} \left[\int_a^b (M(x) - K)^2\, dx \right]^{1/2} \right\}^2,$$

or

$$\mathcal{L}^2 = \lambda^2 (B^2 - hM^2) \left\{ 1 + \frac{h^{3/2}|\lambda|}{|D(\lambda)|} [M^2 - hK^2]^{1/2} \right\}^2;$$

$$B^2 = \int_a^b \int_a^b K^2(x, \xi)\, d\xi\, dx; \tag{13}$$

$$M^2 = \frac{1}{h^2} \int_a^b \left(\int_a^b K(x,\xi)\, d\xi \right)^2 dx.$$

Since the functions $\varphi(x)$, $K(x, \xi)$ belong to the space $L^2(a, b)$, it follows from equalities (2) and (4) ($D(\lambda) \neq 0$) that

$$\Phi_1(K) \leq C \qquad (\delta_1(x) = y_1(x)). \tag{14}$$

Let $\mathcal{L}^2 < 1$; then from (12) and (14) it follows that, as $n \to \infty$, $\Phi_n(K) \to 0$; consequently, by (11), $\alpha_n^2 \to 0$, and hence also $\alpha_n \to 0$.

From

$$\Phi_n(K) = \int_a^b (\delta_n(x) - \lambda h K \alpha_n)^2\, dx$$

it follows that, as $n \to \infty$,

$$\int_a^b \delta_n^2(x)\, dx \to 0;$$

i.e. the sequence of functions $y_n(x)$ converges in itself. Since the space $L^2(a,b)$ is complete, it follows that the sequence of functions $y_n(x)$ converges, as $n \to \infty$, to a function $Y(x)$ belonging to the space $L^2(a,b)$. It is evident that the function $Y(x)$ is a solution of equation (1).

The derived condition $\mathcal{L}^2 < 1$ is less restrictive than the condition given in paper (^{2}).

In the cases $K(x,\xi) \equiv C$ and $K(x,\xi) \equiv K(x)$, $B^2 - hM^2 = 0$, $\mathcal{L}^2 = 0$, $\alpha_2 = 0$; the first approximation gives the exact solution.

If

$$\int_a^b K(x,\xi)\, d\xi = 0,$$

then $\mathcal{L}^2 = \lambda^2 B^2$. In this case the method of averaging functional corrections degenerates into the method of successive approximations. For $\lambda^2 B^2 < 1$, as is known, the convergence of the method of successive approximations has been proved.

Example. Consider the simple equation

$$y(x) = -20.2\sqrt{x} + 3\int_0^1 \sqrt{x}\,(\xi + 10) y(\xi)\, d\xi,$$

which has the obvious solution $y(x) = \sqrt{x}$.

For this example we have: $B^2 = \frac{331}{6}$; $\quad M^2 = \frac{441}{8}$; $\quad K^2 = 49$; $\quad D(\lambda) = 1-$

$$-3 \cdot 7 = -20; \qquad \mathcal{L}^2 = \frac{3}{8}\left(1 + \frac{21}{80}\sqrt{2}\right)^2 < 1.$$

It should be noted that the usually employed sufficient condition for convergence of the method of successive approximations is not satisfied in the present case, since $\lambda^2 B^2 = \frac{331}{6} \cdot 9 > 1$. The ordinary process of successive approximations diverges in this example. However, by the method of averaging functional corrections this equation is solved. The first and second approximations have the form

$$y_1(x) = \sqrt{x} + 0.01\sqrt{x};$$

$$y_2(x) = \sqrt{x} - 0.0001\sqrt{x}.$$

$$y_1(x) - y_2(x) = 0.0001\sqrt{x},$$

i.e., the relative error is 0.01%.

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Note: Figure translations are in progress. See original paper for figures.

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