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Abstract

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Yu. A. SURINOV

ON SOME QUESTIONS IN THE THEORY OF RADIATION TRANSFER AND RADIANT HEAT EXCHANGE IN AN ABSORBING MEDIUM

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Let us consider the processes of radiation transfer and radiant heat exchange in a radiating system consisting of bounding gray bodies separated by an absorbing, moving medium containing internal heat sources ([1], [2]). The initial integral equation for the density of the semi-spherical incident radiation E , describing both the fundamental and the mixed formulation of the problem,* will then have the form

$$E\ \ (M) - \int_F \widetilde{R}(N)\, E\ \ (N)\, K(M,N)\, dF_N = \\ = \int_V \eta_c(P)\, L(M,P)\, dV_P + \int_F E_a(N)\, K(M,N)\, dF_N, \quad M \in \mathfrak{A}_F, \tag{1}$$

where η_c is the density of the intrinsic volume radiation**;

$$K(M,N) = e^{-h}\, \frac{\cos\theta_M \cos\theta_N}{\pi r_{MN}^2}; \qquad L(M,P) = e^{-\Delta h}\, \frac{\cos\theta_M}{4\pi r_{MP}^2}; \tag{2}$$

$$h = \int_0^r \alpha(P)\, ds; \qquad \Delta h = \int_{r^*}^r \alpha(P)\, ds,$$

α is the coefficient of volume absorption; h and Δh are optical thicknesses; θ_M and θ_N are the angles between the ray direction and the directions of the normals to the surface elements at the points M and N; r_{MN} and r_{MP} are, respectively, the distances between the points M, N and M, P.

In the case of the mixed formulation of the problem, in equation (1) we have

$$\widetilde{R}(N) = \begin{cases} R(N), & \text{for } N \in \mathfrak{A}_{F_1}, \\ 1, & \text{for } N \in \mathfrak{A}_{F_2}, \end{cases} \qquad E_a(N) = \begin{cases} E_c(N), & \text{for } N \in \mathfrak{A}_{F_1}, \\ -E\ \ (N), & \text{for } N \in \mathfrak{A}_{F_2}. \end{cases} \tag{3}$$

For the fundamental formulation of the problem, in equation (1) one must set

$$\widetilde{R}(N) \equiv R(N); \qquad E_a(N) \equiv E_c(N).$$

Integral equation (1) is obtained as a result of substituting into the expression for E ([1], [2]) the value E , determined by the relation

$$E\ \ (N) = \widetilde{R}(N)\, E\ \ (N) + E_a(N), \tag{4}$$

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* The fundamental formulation of the problem of radiant heat exchange is characterized by the specification, along with the configuration of the system, of the temperature fields and optical constants both on the boundary of the system F and throughout the volume V of the medium filling it. In the case of the mixed formulation of the problem, the boundary surface is divided into two regions F_1 and F_2 ($F = F_1 + F_2$), and for the region F_2, instead of the temperature, the normal projection of the spherical radiation vector is specified (the density of the resultant radiation E).

** In what follows, everywhere, except in cases that will be specially specified, all quantities will be regarded as spectral, referred to the given frequency ν.

which, for conditions (2), combines the usual relations

$$E_{\text{eff}} = RE_{\text{inc}} + E_c = E_{\text{inc}} - E_{\text{res}}. \tag{5}$$

The existence and uniqueness of the solution of equation (1) (and of some of its special cases that are important from the theoretical and applied points of view) follow from the fact that the corresponding homogeneous equation has no solutions other than the trivial (or zero) one ([3,4]), which is easily proved using the inequality

$$\int_F \widetilde{R}(N) K(M,N)\, dF_N < \int_F K(M,N)\, dF_N = 1 - 4\int_V \alpha(P) L(M,P)\, dV_p < 1. \tag{6}$$

Solving equation (1) by the method of iterations, we obtain the expression

$$E_{\text{inc}}(M) = \int_F \Gamma(M,N) E_a(N)\, dF_N + \int_V Z(M,P) \eta_c(P)\, dV_p, \qquad M \in \mathfrak{A}_V, \tag{7}$$

or

$$\begin{aligned} E_{\text{inc}}(M) = \frac{E_{\text{res}}(M) + E_c(M)}{A(M)} &= \int_{F_1} \Gamma(M,N_1) E_c(N_1)\, dF_{N_1} \\ &- \int_{F_2} \Gamma(M,N_2) E_{\text{res}}(N_2)\, dF_{N_2} + \int_V Z(M,P)\eta_c(P)\, dV_p. \end{aligned} \tag{8}$$

($\mathfrak{A}_V, \mathfrak{A}_F, \mathfrak{A}_{F_1}$ and $\mathfrak{A}_{F_2}$ are sets of points in the volume V and on the surfaces F, F_1 and F_2 ([1,2])), where*

$$\begin{aligned} \Gamma(M,N) - K(M,N) &= \int_F \widetilde{R}(P) \Gamma(M,P) K(P,N)\, dF_p \\ &= \int_F \widetilde{R}(P) K(M,P) \Gamma(P,N)\, dF_p, \end{aligned} \tag{9}$$

$$\begin{aligned} Z(M,P) - L(M,P) &= \int_F \widetilde{R}(N) \Gamma(M,N) L(N,P)\, dF_N \\ &= \int_F \widetilde{R}(N) K(M,N) Z(N,P)\, dF_N. \end{aligned} \tag{10}$$

Expression (8) makes it possible, in the case of a mixed formulation of the problem, to determine both $E_{\text{res}}(M)$ for $M \in \mathfrak{A}_{F_1}$ and the density of the self-radiation $E_{\text{self}}(M)$ for $M \in \mathfrak{A}_{F_2}$, and, consequently, also the temperature field** for the region F_2. To determine the radiation vector $\mathbf{E}_{4\pi}$ and the spatial density of the incident radiation η_{inc} (or the volumetric density of radiant energy u)

* The integral equations for the resolvents $\Gamma(M,N)$ and $Z(M,P)$ are obtained with the aid of transformations of infinite series of the form:

$$\Gamma(M,N) = K(M,N) + \sum_{n=1}^{\infty} K^{(n)}(M,N), \qquad Z(M,P) = L(M,P) + \sum_{n=1}^{\infty} L^{(n)}(M,P),$$

where

$$K^{(n)}(M,N) = \underbrace{\int_F \cdots \int_F}_{n} \widetilde{R}(P_1) \cdots \widetilde{R}(P_n) K(M,P_1) K(P_1,P_2) \cdots K(P_n,N)\, dF_{p_1} \cdots dF_{p_n},$$

$$L^{(n)}(M,P) = \underbrace{\int_F \cdots \int_F}_{n} \widetilde{R}(M_1)\cdots\widetilde{R}(M_n)K(M,M_1)\cdots K(M_{n-1},M_n)L(M_n,P)\,dF_{M_1}\cdots dF_{M_n}.$$

** To determine the temperature it is necessary, however, first to carry out subordinate integration of expression (10) over the entire frequency spectrum.

we obtain the expressions

$$\eta_{\text{inc}}(M) = cu(M) = \int_F \Gamma_1(M,N)E_a(N)\,dF_N + \int_V Z_1(M,P)\eta_c(P)\,dV_p, \quad M \in \mathfrak{A}_V; \tag{11}$$

$$\mathbf{E}_{4\pi}(M) = \int_F {}_*(M,N)E_a(N)\,dF_N + \int_V \mathbf{Z}_*(M,P)\eta_c(P)\,dV_p, \quad M \in \mathfrak{A}_V^*; \tag{12}$$

where

$$\Gamma_1(M,N) = K_1(M,N) + \int_F \widetilde{R}(P)K_1(M,P)\Gamma(P,N)\,dF_p; \tag{13}$$

$$Z_1(M,P) = L_1(M,P) + \int_F \widetilde{R}(N)K_1(M,N)Z(N,P)\,dF_N; \tag{14}$$

$${}_*(M,N) = \int_F \widetilde{R}(P)\mathbf{K}_1(M,P)\Gamma(P,N)\,dF_p = \mathbf{K}_1(M,N) = \mathbf{r}_1 K_1(M,N); \tag{15}$$

$$\mathbf{Z}_*(M,P) = \int_F \widetilde{R}(N)\mathbf{K}_1(M,N)Z(N,P)\,dF_N = \mathbf{L}_1(M,P) = \mathbf{r}_1 L_1(M,P), \tag{16}$$

where $\mathbf{r}_1$ is the unit vector directed along the axis s of the elementary solid angle $d\omega$; c is the speed of light and

$$K_1(M,N) = e^{-h}\frac{\cos\theta_N}{\pi r_{MN}^2}, \qquad L_1(M,P) = e^{-\Delta h}\frac{1}{4\pi r_{MP}^2}. \tag{17}$$

One of the principal characteristics of the radiation field is, along with $\mathbf{E}_{4\pi}$ and η_{inc}, the volume density of the resultant radiation η_{res}, defined by the relation

$$\eta_{\text{res}} = \alpha \eta_{\text{inc}} - \eta_c. \tag{18}$$

Let us note that this relation is always valid, independently of the properties of the medium and the character of the problem and, in particular, both for a purely absorbing medium and for an absorbing and scattering medium, and in the case of both a stationary and a nonstationary problem.

However, the form of the expanded expression for η_{res}, obtained as a result of substituting into (18) the value of η_{inc} according to an expression of type (11), will naturally depend on the circumstances indicated above. Carrying out such a substitution in (18) of the value of η_{inc} according to (11), we obtain**

$$\begin{aligned}\eta_{\text{res}}(M) &= -\left(\operatorname{div}\mathbf{E}_{4\pi} + \frac{\partial u}{\partial \tau}\right) \\ &= \alpha(M)\int_F \Gamma_1(M,N)E_a(N)\,dF_N + \\ &\quad + \alpha(M)\int_V Z_1(M,P)\eta_c(P)\,dV_p - \eta_c(M), \qquad M \in \mathfrak{A}_V.\end{aligned} \tag{19}$$

The results obtained are the basis of the theory of the thermal-radiation field in purely absorbing media bounded by partially reflecting diffuse surfaces of arbitrary configuration. With regard to

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Considering the case of thermodynamic equilibrium in a radiating system on the basis of the second law of thermodynamics, from (7), (11), and (12) we obtain the following optico-geometrical closure equations (5):

$$\int_{F_1} A(N_1)\Gamma(M,N_1)\,dF_{N_1} + 4\int_V \alpha(P)Z(M,P)\,dV_p = 1, \qquad M \in \mathfrak{A}_F; \tag{7a}$$

$$\int_{F_1} A(N_1)\Gamma_1(M,N_1)\,dF_{N_1} + 4\int_V \alpha(P)Z_1(M,P)\,dV_p = 1, \qquad M \in \mathfrak{A}_V; \tag{11a}$$

$$\int_{F_1} A(N_1)\,{}_*(M,N_1)\,dF_{N_1} + 4\int_V \alpha(P)\mathbf{Z}_*(M,P)\,dV_p = 0, \qquad M \in \mathfrak{A}_V. \tag{12a}$$

The first two of these equations are scalar, and the third equation is vectorial.

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See the footnote on p. 824.

of the boundary surfaces it is assumed that they are Lyapunov surfaces ([4,5]).

The expressions given above, strictly speaking, are valid only for the stationary problem, since in their derivation the radiation-transfer equation without the nonstationary term $\frac{1}{c}\frac{\partial B}{\partial \tau}$ ([1,2]) was used. However, in view of the smallness of this term under ordinary conditions, it may be neglected. In this approximation the solutions obtained above may also be regarded as valid for the nonstationary problem. In this case, for a moving medium containing internal heat sources of productivity q, we obtain the following generalized energy equation ([6])

$$\left[\rho\frac{di}{d\tau} - \operatorname{div}(\lambda \operatorname{grad} T) - q - \frac{dP}{d\tau} - \mu \operatorname{Diss} f(w)\right]_M = \\ = \int_0^\infty \alpha_\nu(M) \int_V Z_1(M,P)\, \eta_{c,\nu}(P)\, dV_p\, d\nu \\ + \int_0^\infty \alpha_\nu(M) \int_F \Gamma_1(M,N) E_{a,\nu}(N)\, dF_N\, d\nu - \eta_c(M), \tag{20}$$

where the term $\mu \operatorname{Diss} f(w)$ characterizes energy dissipation; the spectral quantities here are supplied with the index ν.

Equation (20) is a nonlinear integro-differential equation for the temperature field of the medium, taking into account the boundary conditions for processes of radiant heat exchange. Reductions of this equation to various special cases of combined problems on radiant-conductive and radiant-convective heat exchange, etc., are possible. In an analogous way a generalized energy equation can also be constructed for the more general case of a radiating system filled with an absorbing and scattering medium with an arbitrary scattering indicatrix.

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** Using equations (7a) and (11a), expressions (8) and (19) can be put in the form ([5])

$$
\begin{aligned}
E_{\text{res}}(M) = A(M) \int_{F_1} & A(N_1)\Gamma(M, N_1) \left[E_0(N_1) - E_0(M)\right] dF_{N_1} \\
& - A(M) \int_{F_2} \Gamma(M, N_2) E_{\text{res}}(N_2) \, dF_{N_2} \\
& + 4A(M) \int_V \alpha(P) Z(M, P) \left[E_0(P) - E_0(M)\right] dV_p; \qquad M \in \mathfrak{A}_F;
\end{aligned}
\tag{8a}
$$

$$
\begin{aligned}
\eta_{\text{res}}(M) = \alpha(M) \int_{F_1} & A(N_1)\Gamma_1(M, N_1) \left[E_0(N_1) - E_0(M)\right] dF_{N_1} \\
& - \alpha(M) \int_{F_2} \Gamma(M, N_2) E_{\text{res}}(N) \, dF_{N_2} \\
& + 4\alpha(M) \int_V \alpha(P) Z_1(M, P) \left[E_0(P) - E_0(M)\right] dV_p, \qquad M \in \mathfrak{A}_V.
\end{aligned}
\tag{19a}
$$

Note: Figure translations are in progress. See original paper for figures.

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