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Abstract

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MATHEMATICS

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APPLICATION OF A FUNCTIONAL SERIES FOR DERIVING FORMULAS OF VARIOUS NUMERICAL METHODS FOR SOLVING ORDINARY DIFFERENTIAL EQUATIONS

(Presented by Academician S. L. Sobolev, January 13, 1958)

The works of Sh. E. Mikeladze ([1]), I. S. Mukhina ([2]), and V. P. Vetchinkin ([3]) made it possible, proceeding from the general principles chosen by them, to obtain broad groups of formulas for the numerical integration of ordinary differential equations. However, these results do not encompass the entire variety of possible ways of solving the problem posed.

In this connection it seems expedient to consider the application, for deriving formulas of numerical methods for solving ordinary differential equations, of the following functional series:

$$y(x_n + h) = \sum_{k=0}^{p} \sum_{i=1}^{m_k} h^k C_{ki} y^{(k)}(x_n + \alpha_{ki} h), \qquad m_k = m_0,\ m_1,\ m_2, \dots,\ m_p, \quad (1)$$

where p is the order of the highest derivative of the sought function participating in the computations; m_k is the number of derivatives of the same order participating in the computations.

Formulas for the numerical integration of ordinary differential equations are represented in the form of series (1), whose coefficients are determined from a system of equations obtained most simply by equating the sums of the coefficients of derivatives of the same order after expanding the terms of series (1) in powers and the coefficients of the corresponding derivatives of the first terms of the Taylor series.

The system of equations determining the values of the coefficients of series (1) can be represented in the form

$$\sum_{k=0}^{l-1}\sum_{i=1}^{m_k} C_{ki}\frac{\alpha_{ki}^{l-k-1}}{(l-k-1)!}=\frac{1}{(l-1)!},$$

$$l=1,2,\dots,\ m_0+m_1+m_2+\dots+m_p,\dots,q, \tag{2}$$

where q is the greatest number of equations for which it is possible to find a solution of system (2).

Thus, the problem of deriving formulas for the numerical integration of ordinary differential equations is reduced to the solution of a system of algebraic equations, which, for chosen coefficients α_{ki}, is a linear system.

The investigation of system (2) for determining the values of the coefficients α_{ki}, with the aim of obtaining the most effective formulas for the numerical integration of ordinary differential equations, can be carried out with the aid of a high-speed computing machine.

As an example, a system of equations of the form (2) has been compiled, determining the coefficients of series (1) for $m_0 = m_1 = m_2 = 4$. As an application of series (1), Table 1 has been compiled, in which, along with previously known formulas, pri-

Table 1

C_{01}	C_{11}	C_{12}	C_{13}	C_{21}	α_{01}	α_{11}	α_{12}	α_{13}	α_{21}	Expression for the approximation and rounding error at a step	Distortion coefficients k_i for the given i	Authors of formulas
1	$\frac{23}{12}$	$-\frac{16}{12}$	$\frac{5}{12}$		0	0	−1	−2		$\delta_{n+1} = \delta_n + \frac{h}{12}(23\delta'_n - 16\delta'_{n-1} + 5\delta'_{n-2}) + \sum_i^{i=4,5,\dots} y_n^{(i)} \frac{h^i}{i!}(k_i - 1)$; $k_i = \frac{i}{12}\left[5(-2)^{i-1} - 16(-1)^{(i-1)}\right]$	$-8;\ \frac{80}{3};\ -72;\ \frac{532}{3}$	Adams
1	$\frac{8}{12}$	$-\frac{1}{12}$	$\frac{5}{12}$		0	0	−1	1		$\delta_{n+1} = \delta_n + \frac{h}{12}(8\delta'_n - \delta'_{n-1} + 5\delta'_{n+1}) + \sum_i^{i=4,5,\dots} y_n^{(i)} \frac{h^i}{i!}(k_i - 1)$; $k_i = \frac{i}{12}\left[5 - (-1)^{i-1}\right]$	$2;\ \frac{5}{3};\ 3;\ \frac{7}{3}$	
1	$\frac{8}{3}$	$-\frac{2}{3}$		$-\frac{2}{3}$	−1	0	−1		−1	$\delta_{n+1} = \delta_{n-1} + \frac{h}{3}(8\delta'_n - 2\delta'_{n-1}) - \frac{2h^2}{3}\delta''_{n-1} + \sum_i^{i=4,5,\dots} y_n^{(i)} \frac{h^i}{i!}(k_i - 1)$; $k_i = (-1)^i - \frac{2i}{3}(-1)^{i-1} - \frac{2}{3}i(i-1)(-1)^{i-2}$	$-\frac{13}{3};\ 9;\ -15;\ \frac{67}{3}$	Mukhin

C_{01}	C_{11}	C_{12}	C_{13}	C_{21}	α_{01}	α_{11}	α_{12}	α_{13}	α_{21}	Expression for the approximation and rounding error at a step	Distortion coefficients k_i for the given i	Authors of formulas
1	$\frac{4}{6}$	$\frac{2}{6}$		$\frac{5}{6}$	0	0	−1		0	$\delta_{n+1} = \delta_n + \frac{h}{6}(4\delta'_n + 2\delta'_{n-1}) + \frac{5h^2}{6}\delta''_n + \sum_i^{i=4,5,\dots} y_n^{(i)} \frac{h^i}{i!}(k_i - 1)$; $k_i = \frac{2i}{6}(-1)^{i-1}$	$\frac{4}{3}$; $\frac{5}{3}$; -2; $\frac{7}{3}$	
1	$\frac{2}{3}$	$\frac{1}{3}$		$\frac{1}{6}$	0	0	1		0	$\delta_{n+1} = \delta_n + \frac{h}{3}(2\delta'_n + \delta'_{n+1}) + \frac{h^2}{6}\delta''_n + \sum_i^{i=4,5,\dots} y_n^{(i)} \frac{h^i}{i!}(k_i - 1)$; $k_i = \frac{i}{3}$	$\frac{4}{3}$; $\frac{5}{3}$; 2; $\frac{7}{3}$	Doffing
1	$\frac{8}{3}$	$-\frac{4}{3}$	$\frac{8}{3}$		−3	0	−1	−2		$\delta_{n+1} = \delta_{n-3} - \frac{h}{3}(8\delta'_n - 4\delta'_{n-1} + 8\delta'_{n-2}) + \sum_i^{i=5,6,\dots} y_n^{(i)} \frac{h^i}{i!}(k_i - 1)$; $k_i = (-3)^i + \frac{4i}{3}[2(-2)^{i-1} - (-1)^{i-1}]$	$-\frac{109}{3}$; 225; $-\frac{3005}{3}$; 3841	Milne

C_{01}	C_{11}	C_{12}	C_{13}	C_{21}	α_{01}	α_{11}	α_{12}	α_{13}	α_{21}	Expression for the approximation and rounding error at a step	Distortion coefficients k_i for the given i	Authors of formulas
1	$\frac{4}{3}$	$\frac{1}{3}$	$\frac{1}{3}$		-1	0	-1	1		$\delta_{n+1} = \delta_{n-1} + \frac{h}{3}(4\delta'_n + \delta'_{n-1} + \delta'_{n+1}) + \sum_i^{i=5,6,\dots} y_n^{(i)} \frac{h^{(i)}}{i!}(k_i - 1)$; $k_i = (-1)^i + \frac{i}{3}\left[1 + (-1)^{i-1}\right]$	$\frac{7}{3}$; 1; $\frac{11}{3}$; 1	Simpson

System of equations determining the values of the coefficients of the series (1) for the case $m_0 = m_1 = m_2 = 4$

$$C_{01} + C_{02} + C_{03} + C_{04} + 0 + 0 + 0 + 0 + 0 + 0 + 0 + 0 = 1,$$
$$C_{01}\alpha_{01} + C_{02}\alpha_{02} + C_{03}\alpha_{03} + C_{04}\alpha_{04} + C_{11} + C_{12} + C_{13} + C_{14} + 0 + 0 + 0 + 0 = 1,$$
$$\frac{C_{01}\alpha_{01}^2}{2!} + \frac{C_{02}\alpha_{02}^2}{2!} + \frac{C_{03}\alpha_{03}^2}{2!} + \frac{C_{04}\alpha_{04}^2}{2!} + C_{11}\alpha_{11} + C_{12}\alpha_{12} + C_{13}\alpha_{13} + C_{14}\alpha_{14} + C_{21} + C_{22} + C_{23} + C_{24} = \frac{1}{2!},$$
$$\frac{C_{01}\alpha_{01}^3}{3!} + \frac{C_{02}\alpha_{02}^3}{3!} + \frac{C_{03}\alpha_{03}^3}{3!} + \frac{C_{04}\alpha_{04}^3}{3!} + \frac{C_{11}\alpha_{11}^2}{2!} + \frac{C_{12}\alpha_{12}^2}{2!} + \frac{C_{13}\alpha_{13}^2}{2!} + \frac{C_{14}\alpha_{14}^2}{2!} + C_{21}\alpha_{21} + C_{22}\alpha_{22} + C_{23}\alpha_{23} +$$
$$\frac{C_{01}\alpha_{01}^4}{4!} + \frac{C_{02}\alpha_{02}^4}{4!} + \frac{C_{03}\alpha_{03}^4}{4!} + \frac{C_{04}\alpha_{04}^4}{4!} + \frac{C_{11}\alpha_{11}^3}{3!} + \frac{C_{12}\alpha_{12}^3}{3!} + \frac{C_{13}\alpha_{13}^3}{3!} + \frac{C_{14}\alpha_{14}^3}{3!} + \frac{C_{21}\alpha_{21}^2}{2!} + \frac{C_{22}\alpha_{22}^2}{2!} + \frac{C_{23}\alpha_{23}^2}{2!}$$
$$\frac{C_{01}\alpha_{01}^5}{5!} + \frac{C_{02}\alpha_{02}^5}{5!} + \frac{C_{03}\alpha_{03}^5}{5!} + \frac{C_{04}\alpha_{04}^5}{5!} + \frac{C_{11}\alpha_{11}^4}{4!} + \frac{C_{12}\alpha_{12}^4}{4!} + \frac{C_{13}\alpha_{13}^4}{4!} + \frac{C_{14}\alpha_{14}^4}{4!} + \frac{C_{21}\alpha_{21}^3}{3!} + \frac{C_{22}\alpha_{22}^3}{3!} + \frac{C_{23}\alpha_{23}^3}{3!}$$
$$\frac{C_{01}\alpha_{01}^6}{6!} + \frac{C_{02}\alpha_{02}^6}{6!} + \frac{C_{03}\alpha_{03}^6}{6!} + \frac{C_{04}\alpha_{04}^6}{6!} + \frac{C_{11}\alpha_{11}^5}{5!} + \frac{C_{12}\alpha_{12}^5}{5!} + \frac{C_{13}\alpha_{13}^5}{5!} + \frac{C_{14}\alpha_{14}^5}{5!} + \frac{C_{21}\alpha_{21}^4}{4!} + \frac{C_{22}\alpha_{22}^4}{4!} + \frac{C_{23}\alpha_{23}^4}{4!}$$
$$\frac{C_{01}\alpha_{01}^7}{7!} + \frac{C_{02}\alpha_{02}^7}{7!} + \frac{C_{03}\alpha_{03}^7}{7!} + \frac{C_{04}\alpha_{04}^7}{7!} + \frac{C_{11}\alpha_{11}^6}{6!} + \frac{C_{12}\alpha_{12}^6}{6!} + \frac{C_{13}\alpha_{13}^6}{6!} + \frac{C_{14}\alpha_{14}^6}{6!} + \frac{C_{21}\alpha_{21}^5}{5!} + \frac{C_{22}\alpha_{22}^5}{5!} + \frac{C_{23}\alpha_{23}^5}{5!}$$
$$\frac{C_{01}\alpha_{01}^8}{8!} + \frac{C_{02}\alpha_{02}^8}{8!} + \frac{C_{03}\alpha_{03}^8}{8!} + \frac{C_{04}\alpha_{04}^8}{8!} + \frac{C_{11}\alpha_{11}^7}{7!} + \frac{C_{12}\alpha_{12}^7}{7!} + \frac{C_{13}\alpha_{13}^7}{7!} + \frac{C_{14}\alpha_{14}^7}{7!} + \frac{C_{21}\alpha_{21}^6}{6!} + \frac{C_{22}\alpha_{22}^6}{6!} + \frac{C_{23}\alpha_{23}^6}{6!}$$
$$\frac{C_{01}\alpha_{01}^9}{9!} + \frac{C_{02}\alpha_{02}^9}{9!} + \frac{C_{03}\alpha_{03}^9}{9!} + \frac{C_{04}\alpha_{04}^9}{9!} + \frac{C_{11}\alpha_{11}^8}{8!} + \frac{C_{12}\alpha_{12}^8}{8!} + \frac{C_{13}\alpha_{13}^8}{8!} + \frac{C_{14}\alpha_{14}^8}{8!} + \frac{C_{21}\alpha_{21}^7}{7!} + \frac{C_{22}\alpha_{22}^7}{7!} + \frac{C_{23}\alpha_{23}^7}{7!}$$
$$\frac{C_{01}\alpha_{01}^{10}}{10!} + \frac{C_{02}\alpha_{02}^{10}}{10!} + \frac{C_{03}\alpha_{03}^{10}}{10!} + \frac{C_{04}\alpha_{04}^{10}}{10!} + \frac{C_{11}\alpha_{11}^9}{9!} + \frac{C_{12}\alpha_{12}^9}{9!} + \frac{C_{13}\alpha_{13}^9}{9!} + \frac{C_{14}\alpha_{14}^9}{9!} + \frac{C_{21}\alpha_{21}^8}{8!} + \frac{C_{22}\alpha_{22}^8}{8!} + \frac{C_{23}\alpha_{23}^8}{8!}$$
$$\frac{C_{01}\alpha_{01}^{11}}{11!} + \frac{C_{02}\alpha_{02}^{11}}{11!} + \frac{C_{03}\alpha_{03}^{11}}{11!} + \frac{C_{04}\alpha_{04}^{11}}{11!} + \frac{C_{11}\alpha_{11}^{10}}{10!} + \frac{C_{12}\alpha_{12}^{10}}{10!} + \frac{C_{13}\alpha_{13}^{10}}{10!} + \frac{C_{14}\alpha_{14}^{10}}{10!} + \frac{C_{21}\alpha_{21}^9}{9!} + \frac{C_{22}\alpha_{22}^9}{9!} + \frac{C_{23}\alpha_{23}^9}{9!}$$

new ones have been introduced. In writing the expressions for the approximation errors and the round-off errors at a step, which occur when using the formulas of Table 1, the notation $\delta = y - \bar{y}$, $\delta' = y' - \bar{y}'$, $\delta'' = y'' - \bar{y}''$ has been adopted (bars are placed over the exact values of the sought function and its derivatives).

Formulas expressed in terms of finite numbers of equidistant ordinates, as is known, can be transformed into formulas expressed in terms of differences.

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Note: Figure translations are in progress. See original paper for figures.