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Abstract

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MATHEMATICS

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ADDITION OF A PRIME NUMBER TO A PRIME POWER OF A GIVEN PRIME

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§ 1. In the present paper we consider certain additive problems of binary type with prime numbers. In particular, we solve the question of the number of integers n, not exceeding a given bound, for which the equation

$$n = p_1 + p_2^{p_3} \tag{1}$$

is soluble in primes p_1, p_2, p_3, where p_2 is fixed.

Theorem 1. "Almost" all numbers of the sequence $p_1 + p_2^{p_3}$, where p_1, p_2 independently run through prime numbers and p_2 is a fixed prime, are distinct.

In other words, the number of integers representable in the form (1) in more than one way has a smaller order of growth in comparison with the number of integers representable in the form (1), but in only one way.

A more precise fact is supplied by the following theorem.

Theorem 2. Let $Q(p_2, N)$ be the number of all integers $n \leqslant 2N$ representable in the form (1) with $p_1 \leqslant N$, $p_2^{p_3} \leqslant N$; let $F(p_2, N)$ be the number of those among them which are representable in only one way. The symbol $\sim$ denotes the sign of asymptotic equality.

Then we have:

$$Q(p_2, N) = \frac{N}{\ln p_2 \cdot \ln\ln N} + O\left(\frac{N \ln\ln\ln N}{\ln p_2 \cdot \ln^2 \ln N}\right),$$

$$F(p_2, N) \sim Q(p_2, N). \tag{2}$$

Let us note here that the sequence $p_1 + p_2^{p_3}$ differs essentially from the sequence $p_1 + p_2^m$, $m = 1, 2, \ldots$, studied in papers ([1],[3],[4]), and for this latter sequence, generally speaking, theorems analogous to Theorems 1 and 2 do not hold.

§ 2. We outline the proof of Theorem 2. Denoting by $R(n, p_2)$ the number of solutions of equation (1), we estimate the quantities $Q(p_2, N)$ and $F(p_2, N)$ from below. We have

$$Q(p_2, N) \geqslant F(p_2, N) = \sum_{n \leqslant 2N} R(n, p_2) - \sum_{\substack{n \leqslant 2N \\ R(n,p_2) \geqslant 2}} R(n, p_2) \geqslant$$

$$\geqslant \sum_{n \leqslant 2N} R(n, p_2) - \sum_{n \leqslant 2N} R(n, p_2)\{R(n, p_2) - 1\}. \tag{3}$$

Let $S(n)$ be equal to the number of solutions of the equation $n = p_i - p_j$ in prime numbers p_i, p_j, under the condition $p_i \leqslant N$, $p_j \leqslant N$, and let us use identity (2) of the paper of N. P. Romanov ([1]). This identity in the present case can be

can be written in the form

$$\sum_{n \leqslant 2N} R^2(n, p_2) = \sum_{n \leqslant 2n} R(n, p_2) + 2 \sum_{1 < n = p_2^{q_1} - p_2^{q_2} \leqslant N} S(n), \tag{4}$$

where q_1, q_2 run through the prime numbers.

From (3) and (4) it now follows that

$$Q(p_2, N) \geqslant F(p_2, N) \geqslant \sum_{n \leqslant 2N} R(n, p_2) - 2 \sum_{1 < n = p_2^{q_1} - p_2^{q_2} \leqslant N} S(n). \tag{5}$$

In what follows let $c, c_1, c_2, \ldots$ denote certain absolute constants.

We turn to estimates of the sums entering into inequality (5). According to the well-known theorem of Viggo Brun–Schnirelmann ([2]),

$$S(n) < c\,\frac{N}{\ln^2 N}\, f(n), \qquad f(n) = \prod_{p \mid n} \left(1 + \frac{1}{p}\right), \tag{6}$$

where p runs through the prime divisors of n.

On the basis of estimate (6) we obtain

$$\sum_{1 < n = p_2^{q_1} - p_2^{q_2} \leqslant N} S(n) < c\,\frac{N}{\ln^2 N} \sum_{q_2 < q_1 \leqslant \ln N / \ln p_2} f(p_2^{q_1} - p_2^{q_2}) = \tag{7}$$

$$= c\,\frac{N}{\ln^2 N}\, f(p_2) \sum_{q_2 < q_1 \leqslant \ln N / \ln p_2} f(p_2^{q_1 - q_2} - 1) \leqslant \frac{3}{2}\, c\,\frac{N}{\ln^2 N} \sum_{q_2 < q_1 \leqslant \ln N / \ln p_2} f(p_2^{q_1 - q_2} - 1)\,.$$

Next put $h = q_1 - q_2$, where $q_2 < q_1$ are prime numbers and

$$q_2, q_1 \leqslant \ln N / \ln p_2. \tag{8}$$

Then for every h satisfying $1 \leqslant h \leqslant \ln N / \ln p_2$, again applying the result of Viggo Brun' s "sieve" —estimate (6), we find that the number of solutions of equation (8) does not exceed

$$c \frac{\ln N}{\ln p_2 \cdot \ln^2 \frac{\ln N}{\ln p_2}} \left\{ \max_{h \leqslant \ln N / \ln p_2} \prod_{p|h} \left(1 + \frac{1}{p}\right) \right\} < c_2 \frac{\ln N \cdot \ln\ln\ln N}{\ln p_2 \cdot \ln^2 \frac{\ln N}{\ln p_2}}.$$

Therefore from (7) we obtain

$$\sum_{1 < n = p_2^{q_1} - p_2^{q_2} \leqslant N} S(n) < c_0 \frac{N \ln\ln\ln N}{\ln N \cdot \ln p_2 \cdot \ln^2 \frac{\ln N}{\ln p_2}} \sum_{1 \leqslant h \leqslant \ln N / \ln p_2} f(p_2^h - 1). \tag{9}$$

To estimate the last sum we introduce the notation: $\delta(k, p_2)$ is the exponent to which the number p_2 belongs modulo k, $\mu(k)$ is the Möbius function, $\varphi(k)$ is Euler' s function, $l = \delta(d, p_2)$.

We now get

$$\sum_{1 \leqslant h \leqslant \ln N / \ln p_2} f(p_2^h - 1) < c_4 \sum_{1 \leqslant k \leqslant N} \frac{\mu^2(k)}{k} \sum_{\substack{1 \leqslant h \leqslant \ln N / \ln p_2 \\ p_2^h \equiv 1 \pmod{k}}} 1 < c_5 \frac{\ln N}{\ln p_2} \sum_{1 \leqslant k \leqslant N} \frac{\mu^2(k)}{k\, \delta(k, p_2)}.$$

The series on the right, as N. P. Romanov ([1]) proved, converges. Moreover, if one puts

$$\sigma(l, p_2) = \sum_{p_2^l \equiv 1 \pmod{d},\ l | \varphi(d)} \frac{\mu^2(d)}{d},$$

then the following inequalities hold:

$$\sum_{1 \leq k \leq N} \frac{\mu^2(k)}{k \delta(k, p_2)} < c_6 \sum_{l=1}^{\infty} \frac{\sigma(l, p_2)}{l} < c_7 \ln^2 \ln 2p_2.$$

The proof of the last of these inequalities is rather complicated and can be obtained by the method of paper ([1]).

Thus we find:

$$\sum_{1\le h\le \ln N/\ln p_2} f\left(p_2^h-1\right) < c_8\frac{\ln N\cdot \ln^2 \ln 2p_2}{\ln p_2}.$$

Combining this estimate with inequality (9), we obtain

$$\sum_{1<n-p_2^{q_1}-p_2^{q_2}\le N} S(n) < c_9\frac{N\ln\ln\ln N\cdot \ln^2\ln 2p_2}{\ln^2\frac{\ln N}{\ln p_2}\cdot \ln^2 p_2}. \tag{10}$$

To estimate the first sum in (5), introduce the notation: $\pi(x)$ is the number of primes $\le x$, and $P(x)$ is the number of numbers $p_2^{p_3}\le x$.

Applying the prime number theorem, we shall have

$$\sum_{n\le 2N} R(n,p_2)=\pi(N)P(N)=\frac{N}{\ln p_2\cdot \ln\frac{\ln N}{\ln p_2}}+O\left(\frac{N}{\ln p_2\cdot \ln N}\right). \tag{11}$$

Collecting the estimates (5), (10), and (11), we obtain

$$F(p_2,N)\ge \frac{N}{\ln p_2\cdot \ln\frac{\ln N}{\ln p_2}}+O\left(\frac{N\ln\ln\ln N}{\ln p_2\cdot \ln^2\frac{\ln N}{\ln p_2}}\right). \tag{12}$$

On the other hand, obviously we have

$$F(p_2,N)\le Q(p_2,N)\le \sum_{n\le 2N} R(n,p_2), \tag{13}$$

so that from (10)–(13) we finally obtain formula (2). Theorem 2 is thereby proved.

§ 3. Similar considerations also make it possible to obtain the following nontrivial generalization of the results of paper ([4]).

Theorem 3. Let $J_k(a,N)$ be the number of numbers $n\le 2N$ representable in the form

$$n=p+a^m, \tag{14}$$

where p is prime; $a\ge 2$ is a given integer; $m=1^k,2^k,\dots;$ $k\ge 2$ is an arbitrary fixed integer; $p\le N;$ $a^m\le N$. Let $G_k(a,N)$ be the number of those numbers which are representable in the form (14) in only one way. Then we have:

$$G_k(a,N)\sim J_k(a,N),$$

$$J_k(a, N) = \frac{N}{(\ln N)^{1-1/k} \ln^{1/k} a} + O\left(\frac{N}{\ln^{1-\varepsilon} N \cdot \ln^{1/k} a}\right),$$

where $\varepsilon > 0$ is an arbitrarily small constant.

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[4] A. F. Lavrik, *Dokl. Akad. Nauk SSSR*, 115, No. 3, 445 (1957).

Note: Figure translations are in progress. See original paper for figures.