Mathematics

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Abstract

Full Text

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THE TRICOMI PROBLEM FOR THE GENERAL LAVRENT' EV-BITSADZE EQUATION

(Presented by Academician I. G. Petrovskii, 19 VI 1957)

1°. Consider the equation

$$L(u) \equiv u_{xx} + \operatorname{sgn} y \, u_{yy} + A(x,y)u_x + B(x,y)u_y + C(x,y)u = 0 \qquad \text{(L)}$$

$$(C(x,y) \leq 0 \text{ for } y > 0).$$

Let the domain D be bounded by a smooth line Γ, lying in the half-plane $y > 0$ and resting on the segment $-1 \leq x \leq 1$ of the axis OX (the segment AB), and by two characteristics AC and BC in the half-plane $y < 0$. The set of points of the domain D for which $y > 0$ $(y < 0)$ will be denoted by D_1 (D_2).

Problem T. Find a function $u(x,y)$ having the following properties:
1) $u(x,y)$ is a solution of equation (L) in each of the domains D_1 and D_2;
2) $u(x,y)$ is continuous in $\overline{D}$;
3) the partial derivatives $\partial u/\partial x$, $\partial u/\partial y$ are continuous in D;
4) on the line Γ and on one of the characteristics, for example AC, $u(x,y)$ assumes prescribed values: $u = \varphi(s)$ on Γ; $u = \psi(x)$ on AC; $\varphi(l) = \psi(-1)$, where l is the length of the curve Γ. The length of the arc s is measured from the point B. The coefficients of equation (L) will be assumed to be twice continuously differentiable functions in each of the domains D_1, D_2, up to their boundaries.

2°. In the domain D_2 we transform equation (L) to characteristic coordinates, setting $\xi = x + y$, $\eta = x - y$. We obtain the equation

$$L_0(u) \equiv u_{\xi\eta} + a(\xi,\eta)u_\xi + b(\xi,\eta)u_\eta + c(\xi,\eta)u = 0. \qquad (L_0)$$

The domain D_2 is transformed into a domain D_0 of the (ξ,η)-plane, which is the triangle $A_0B_0C_0$ with side equations $\xi = \eta$, $\xi = -1$, $\eta = 1$.

We shall assume that the coefficients of equation (L_0) satisfy the following conditions:

Conditions A_1**:** $a \geq 0$, $a_\xi + ab - c \geq 0$, $c \geq 0$ in D_0.

Denote by D^ε_{hk} the triangular domain bounded by: 1) the segment $A_0'B_0'$ of the line $\eta - \xi = \varepsilon$; 2) the segment $A_0'C_0'$ of the line $\xi = h$; 3) the segment $B_0'C_0'$ of the line $\eta = k$ ($\varepsilon \geq 0,\ -1 \leq h < 0 < k \leq 1$).

We shall call a solution $u(\xi, \eta)$ of equation (L_0) **strictly regular** in D^ε_{hk} if: 1) $u(\xi, \eta)$ is twice continuously differentiable in D^ε_{hk}; 2) $u(\xi, \eta)$ is continuous in $\overline{D}^\varepsilon_{hk}$ together with its first-order derivatives.

A **generalized solution of class** S of equation (L_0) in D_0 will mean a function $u(\xi, \eta)$, defined in D_0, which can be represented as the limit in D_0 of a sequence $\{u_n(\xi, \eta)\}$ of solutions strictly regular in $\overline{D}_0$, the convergence being uniform in every closed domain $\overline{D}^\varepsilon_{-1,\,1-\varepsilon}$ ($\varepsilon > 0$), and $u_n(-1, \eta) = u(-1, \eta)$.

Lemma 1. Let $u(\xi, \eta)$ be a generalized solution of equation (L_0) of class S, continuous in $\overline{D}_0$, and suppose that $u(-1, \eta) = 0$. If the conditions A_1 are satisfied, then $\max u$, if it is > 0, is attained on A_0B_0.

Proof. We note that the lemma is valid for strictly regular solutions ([1]). Suppose, in our case, that $\max u = M > 0$ is attained not on A_0B_0. Then $u = M$ on some closed set Φ not containing points of the segment A_0B_0. There will be found a closed domain $\overline{D}^\alpha_{-1,1-\alpha}$, $\alpha > 0$, and n such that $u_n(\xi, \eta)$ will attain a positive maximum not on $A_0'B_0'$, which contradicts the property of strictly regular solutions.

For what follows we shall also need the following known lemma ([2]).

Lemma 2. Let the function $u(x, y)$, continuous in $\overline{D}_1$, satisfy in $\overline{D}_1$ the inequality $L(u) \geq 0$ ($L(u) \leq 0$), with $C \leq 0$. Suppose $u(x, y)$ assumes its greatest positive (least negative) value in $\overline{D}_1$ at the point $(x_0, 0)$, $-1 < x_0 < 1$. If the values of u on Γ are less (greater) than $u(x_0, 0)$, then

$$\lim_{y \to 0} \frac{\partial u(x_0, y)}{\partial y} < 0 \ (> 0),$$

provided this limit exists.

Denote by $u_0(\xi', \eta'; \xi, \eta)$ the Riemann function of equation (L_0). By a **solution of equation** (L_0) **of class** R **in the characteristic triangle** D_0 we shall mean a function $u(\xi, \eta)$ which in D_0 can be represented by the Riemann formula

$$u(\xi, \eta) = \frac{1}{2}[u_0(\xi, \xi; \xi, \eta)\tau(\xi) + u_0(\eta, \eta; \xi, \eta)\tau(\eta)] + \\ + \int_\xi^\eta M(t, \xi, \eta)\tau(t)\,dt + \int_\xi^\eta N(t, \xi, \eta)\nu(t)\,dt, \qquad (1)$$

where $\tau(x)$ is continuous on $-1 \leq x \leq 1$, differentiable on $-1 < x < 1$, and $\nu(x)$ is continuous on $-1 < x < 1$, absolutely integrable on $-1 \leq x \leq 1$. Here M and N are known functions entering into the Riemann formula.

Lemma 3. Every solution of class R belongs to class S.

From Lemmas 1, 2, and 3 follows the uniqueness of the solution of the problem in class R.

3°. We note that a continuously differentiable in D_0 solution of the equation $L_0(u) = f(\xi, \eta)$, satisfying the conditions $u(-1, \eta) = 0$, $u_\xi(\xi, \xi) - u_\eta(\xi, \xi) = 0$, can be represented by the formula

$$u(\xi, \eta) = \int_{-1}^{\xi} d\xi' \int_{\xi'}^{\xi} f(\xi', \eta') S(\xi', \eta'; \xi, \eta)\, d\eta' + \\ + \int_{-1}^{\xi} d\xi' \int_{\xi'}^{\eta} f(\xi', \eta') T(\xi', \eta'; \xi, \eta)\, d\eta', \tag{2}$$

where S, T are continuous in $\overline{D}_0 \times \overline{D}_0$. Formula (2) and the corresponding expressions for S and T can be obtained, for example, by applying to the solution of the problem the method of successive approximations.

Lemma 4. Every solution of equation (L_0) in D_0 of class R can be represented in the form

$$u(\xi, \eta) = g(\xi, \eta) + \int_{-1}^{\xi} T_0(\xi, \eta, t) \nu(t)\, dt, \tag{3}$$

where $g(\xi, \eta)$ is expressed in terms of ψ, a, b, c, S, T; $T_0(\xi, \eta, t)$ is expressed in terms of a, b, c, S, T; $\nu(\xi) = (\partial u/\partial \xi - \partial u/\partial \eta)_{\xi=\eta}$.

From (3), putting $g(x) = g(x, x)$, $T_0(x, t) = T_0(x, x, t)$, $\tau(x) = u(x, x)$, we find:

$$\tau(x) = g(x) + \int_{-1}^{x} T_0(x, t) \nu(t)\, dt. \tag{4}$$

4°. Problem N for the domain D_1. Find in D_1 a solution of equation (L) satisfying the conditions:

$$u\big|_{\Gamma} = \varphi(s), \qquad \frac{\partial u}{\partial y}\bigg|_{y=0} = \nu(x).$$

Conditions B. Let the curve Γ have continuous curvature, and let the tangents to the curve Γ at the points A and B be parallel to the OY-axis.

Let $G(x', y'; x, y)$ be the Green' s function of the Laplace equation for problem N, whose existence is not hard to prove. Following Lichtenstein' s method, we seek the solution of problem N in the form

$$u(x,y) = w(x,y) - \frac{1}{2\pi}\iint_{D_1} G(x', y'; x, y) V(x', y')\, dx'\, dy',$$

where $w(x, y)$ is the solution of problem N for the Laplace equation; it is expressed by the formula

$$w(x,y) = -\frac{1}{2\pi}\int_{-1}^{1} G(t, 0; x, y)\nu(t)\, dt +$$

$$+\frac{1}{2\pi}\int_{-1}^{1} \frac{\partial}{\partial n} G(x(s), y(s); x, y)\varphi(s)\, ds.$$

To find the function $V(x, y)$, we form an integral equation (similarly to (3), p. 267), whose kernel $K_2(x', y'; x, y)$ is the first iterate of the kernel

$$K(x', y'; x, y) = \frac{1}{2\pi}\left[A(x,y)\frac{\partial G}{\partial x} + B(x,y)\frac{\partial G}{\partial y} + C(x,y)G\right].$$

If 1 is not an eigenvalue of the kernel $K_2(x', y'; x, y)$, then, solving the corresponding integral equation, we shall find $V(x, y)$ and obtain the solution of problem N. Putting in it $y = 0$, we arrive at the formula

$$\tau(x) = -\int_{-1}^{1} \left[G(t,x) + K_0(t,x) + K_{00}(t,x)\right]\nu(t)\, dt +$$

$$+\int_{0}^{l} \left[G^*(s,x) + \overline{K}_0(s,x) + \overline{K}_{00}(s,x)\right]\varphi(s)\, ds, \tag{5}$$

where $\tau(x) = u(x, 0)$; $G(t, x) = \dfrac{1}{2\pi} G(t, 0; x, 0)$; $G^*(s, x) = \dfrac{1}{2\pi}\dfrac{\partial G(x(s), y(s); x, 0)}{\partial n}$; $K_0, \overline{K}_0$ are expressed in terms of G, A, B, C, while $K_{00}, \overline{K}_{00}$, in addition, are also expressed in terms of the resolvent of the kernel $K_2(x', y'; x, y)$.

5°. Solution of problem T. Comparing (4) and (5), we obtain an integral equation of the first kind with the unknown function $\nu(x)$. In order to obtain an equation of the second kind, we differentiate the resulting equation of the first kind with respect to x. As a result we obtain the equation

$$\nu(x) + \frac{1}{\pi}\int_{-1}^{1}\left(\frac{1}{t-x} - \frac{t}{1-xt}\right)\nu(t)\, dt = \int_{-1}^{1} \mathcal{K}(t,x)\nu(t)\, dt + g_0(x). \tag{6}$$

Making estimates of $\partial T_0/\partial x$, $\partial K_0/\partial x$, $\partial K_{00}/\partial x$, and also $\partial G(t,0;x,0)/\partial x - \partial G_0(t,0;x,0)/\partial x$, where G_0 is the Green' s function for the case of a circle, we see that the function $\mathcal{K}(t,x)$ has singularities no higher than of logarithmic order. We shall seek the function $\nu(x)$ in the class of functions satisfying the Hölder condition on $-1 < x < 1$, absolutely integrable with some power $\alpha > 1$. Then the first term of the right-hand side will be a function satisfying the Hölder condition on $-1 \le x \le 1$. Assuming the function $\varphi(s)$ to have continuous second derivatives, at least in neighborhoods of the points A and B, and that $\psi'(x)$ satisfies the Hölder condition, we arrive at the conclusion that $g_0(x)$ satisfies the Hölder condition on $-1 \le x \le 1$. Applying the known methods of regularization (see, for example, (5)), we arrive at the integral equation

$$\nu(x) = \int_{-1}^{1} \left\{ \mathcal{K}(t,x) - \frac{1}{2\pi}\sqrt{\frac{1+x}{1-x}} \int_{-1}^{1} \sqrt{\frac{1-u}{1+u}} \left(\frac{1}{u-x} - \frac{u}{1-xu} \right) \mathcal{K}(t,u)\,du \right\} \nu(t)\,dt +$$

$$+ g_0(x) - \frac{1}{2\pi}\sqrt{\frac{1+x}{1-x}} \int_{-1}^{1} \sqrt{\frac{1-t}{1+t}} \left(\frac{1}{t+x} - \frac{t}{1-xt} \right) g_0(t)\,dt. \qquad (7)$$

This is a Fredholm equation, and the solvability of equation (7) follows from the uniqueness of the solution of problem T. Having found $\nu(x)$ from (7), we find $\tau(x)$ from (5), and then find the desired solution in D_1 and D_2. This will be the solution of problem T of class R. Thus, we have:

Theorem 1. Let the contour Γ satisfy condition B, and let the coefficients of equation (L) be such that, for $y < 0$, conditions A_1 are fulfilled and 1 is not an eigenvalue of the kernel $K_2(x', y'; x, y)$. If the function $\varphi(s)$ is continuous on $0 \le s \le l$ together with its second-order derivatives, and $\psi(x)$ has a first derivative satisfying the Hölder condition on $-1 \le x \le 0$, then in the class R there exists, moreover a unique, solution of problem T.

6°. Particular case. Consider the equation

$$u_{xx} + \operatorname{sgn} y \cdot u_{yy} + C(x,y)u = 0 \qquad (C \le 0 \text{ for } y > 0). \qquad (\mathrm{F})$$

If $C(x,y)$ depends only on x or only on y, then problem T for equation (F) always has a solution, provided the above-stated conditions imposed on the contour Γ and on the functions $\varphi(s)$ and $\psi(x)$ are fulfilled.

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.