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Abstract

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MATHEMATICS

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PROOF OF CERTAIN ASYMPTOTIC FORMULAS FOR SOLUTIONS OF DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER

In the paper ⁽¹⁾, formal asymptotic expansions were calculated for the solutions of a system of differential equations with a small parameter

$$\begin{aligned} \varepsilon \dot{x}^i &= f^i(x^1, \dots, x^k, y^1, \dots, y^l), \\ \dot{y}^j &= g^j(x^1, \dots, x^k, y^1, \dots, y^l), \\ i &= 1, \dots, k, \quad j = 1, \dots, l, \end{aligned} \quad (1)$$

in a neighborhood of a "breakdown point," i.e., a point where $\det \|\partial f^i / \partial x^\alpha\| = 0$. These expansions (see formulas (1.50), (2.30), and (3.5) of ⁽¹⁾) were then used substantially both in the paper ⁽¹⁾ itself and in the paper ⁽²⁾. However, in ⁽¹⁾ no proof was given that the formal expansions calculated there do indeed approximate the true solutions of system (1) with the stated accuracy. Here we give a scheme of these proofs, using the notation of ⁽¹⁾.

By a linear change of coordinates, system (1) in a neighborhood of the breakdown point is brought to the form (see § 3 of ⁽¹⁾)

$$\begin{aligned} \varepsilon \dot{\xi}^1 &= (\xi^1)^2 + \eta^1 + b_\beta^1 \eta^\beta + c_\beta^1 \xi^1 \eta^\beta + d_1^1 (\xi^1)^3 + e_\alpha^1 \xi^1 \xi^{\alpha'} + \dots \equiv \Phi^1(\xi, \eta), \\ \varepsilon \dot{\xi}^i &= a_{\alpha'}^i \xi^{\alpha'} + b_\beta^i \eta^\beta + c_0^i (\xi^1)^2 + d_1^i (\xi^1)^3 + e_\alpha^i \xi^1 \xi^{\alpha'} + \dots \equiv \Phi^i(\xi, \eta), \\ \dot{\eta}^j &= \delta_1^j + \alpha_1^j \xi^1 + \dots \equiv \Psi^j(\xi, \eta), \\ i &= 2, \dots, k, \quad j = 1, \dots, l, \end{aligned} \quad (2)$$

where all eigenvalues of the matrix $\|a_{\alpha'}^i\|$ have negative real parts. For $-p \leq \xi^1 \leq p$ (p small, but independent of ε), the quantity ξ^1 may be taken as an independent variable and, instead of system (2), one may consider the system

$$\frac{d\xi^i}{d\xi^1} = \frac{\Phi^i(\xi, \eta)}{\Phi^1(\xi, \eta)}, \quad \frac{d\eta^j}{d\xi^1} = \varepsilon \frac{\Psi^j(\xi, \eta)}{\Phi^1(\xi, \eta)}, \quad (3)$$

$$i = 2, \dots, k, \quad j = 1, \dots, l.$$

The proofs that the formal expansions of the solutions of system (3), found in ⁽¹⁾, represent with a quite definite accuracy the true solutions of this system are carried out differently on each of the three intervals of variation of the variable ξ^1 : $-p \leq \xi^1 \leq -\sigma_1$, $-\sigma_1 \leq \xi^1 \leq \sigma_2$, $\sigma_2 \leq \xi^1 \leq p$, where $\sigma_1 = \varepsilon^{2/7}$, $\sigma_2 = \varepsilon^{1/9}$. However, the main idea of these proofs is the same for all three intervals. This idea consists in constructing a “tube.” The formal approximation is surrounded by a narrow closed neighborhood U , which we call a tube; the diameter of the tube depends on ε and, as $\varepsilon \rightarrow 0$, tends to zero as some positive power of ε .

It is proved that if the initial point of a solution of system (3) is taken in the tube U , then throughout the corresponding interval this solution does not leave the tube. For this purpose the boundary of the tube U is constructed so that some of its “walls” are Lyapunov functions for the system of equations (2), i.e., so that they are intersected by the trajectories of system (2), as t increases in a definite direction, namely “from outside into” the tube.

In constructing the tube we use on all intervals a positive definite quadratic form $W(z^2, \dots, z^k)$ —a Lyapunov function for the linear system

$$\dot{z}^i = d_{\alpha'}^i z^{\alpha'}, \quad i = 2, \dots, k, \quad (4)$$

satisfying the inequality

$$W'_{(4)}(z^2, \dots, z^k) < -\rho W(z^2, \dots, z^k), \quad (5)$$

$\rho > 0$ (see (3)).

1. **The interval** $-p \leq \xi^1 \leq -\sigma_1$. Here new coordinates are introduced by the formulas: $\xi^1 = \xi^1$; $\varphi^i = \Phi^i(\xi, \eta)$, $i = 1, \dots, k$; $\eta^j = \eta^j$, $j = 2, \dots, l$. In these coordinates system (2) is written as

$$\varepsilon \dot{\xi}^1 = \varphi^1, \quad \dot{\eta}^j = G^j(\xi^1, \varphi, \eta, \varepsilon),$$

$$\varepsilon \dot{\varphi}^1 = 2\xi^1 \varphi^1 + K^1(\xi^1, \varphi, \eta, \varepsilon), \quad \varepsilon \dot{\varphi}^i = d_{\alpha'}^i \varphi^{\alpha'} + K^i(\xi^1, \varphi, \eta, \varepsilon). \quad (6)$$

The nonautonomous system obtained from system (6), if in the latter the quantity ξ^1 is taken as the independent variable, will be denoted by (6'). We formally construct the sums: a) $\varphi^{i,2} = \varepsilon \varphi_1^i(\xi^1) + \varepsilon^2 \varphi_2^i(\xi^1)$; b) $\eta^j = \eta_0^j(\xi^1) + \varepsilon \eta_1^j(\xi^1)$, $i = 1, \dots, k$; $j = 2, \dots, l$, where the functions $\varphi_1^i, \varphi_2^i, \eta_0^j, \eta_1^j$ are determined from

relations obtained as a result of substituting the sums a) and b) into system (6') and subsequently equating the coefficients of like powers of ε .

Let us call the **tube** U_1 the set of all points of the space $(\xi^1, \varphi^1, \dots, \varphi^k, \eta^2, \dots, \eta^l)$ whose coordinates satisfy the inequalities:

$$|\varphi^1 - \varphi^{1,2}(\xi^1)| \leq \varepsilon M_1; \quad W(\varphi^2 - \varphi^{2,2}(\xi^1), \dots, \varphi^k - \varphi^{k,2}(\xi^1)) \leq \varepsilon^2 N_1^2;$$

$$|\eta^j - \eta^{j,1}(\xi^1)| \leq \varepsilon P_1, \quad j = 2, \dots, l,$$

where M_1, N_1, P_1 are positive constants independent of ε . The set of points of the tube U_1 singled out by the equation

$$|\varphi^1 - \varphi^{1,2}(\xi^1)| = \varepsilon M_1$$

will be called the φ^1 -wall and denoted by $U_1^{\varphi^1}$; the set of points of the tube U_1 singled out by the equation

$$W(\varphi^2 - \varphi^{2,2}(\xi^1), \dots, \varphi^k - \varphi^{k,2}(\xi^1)) = \varepsilon^2 N_1^2$$

will be called the W -wall and denoted by U_1^W .

Lemma 1. *On the interval $-p \leq \xi^1 \leq -\sigma_1$, the walls $U_1^{\varphi^1}$ and U_1^W of the tube U_1 , for sufficiently large M_1 and N_1 , are surfaces without contact for the system of equations (6), and all trajectories of system (6) starting on the walls $U_1^{\varphi^1}$ and U_1^W , as t increases, enter the tube U_1 .*

For the proof of Lemma 1 one computes the derivative, along system (6), on the walls $U_1^{\varphi^1}$ and U_1^W ; it turns out to be negative.

With the aid of Lemma 1, by the method of successive approximations it is proved that, if the initial point of some solution of system (6') for $\xi^1 = -p$ is taken in the tube U_1 , then throughout the interval $-p \leq \xi^1 \leq -\sigma_1$ this solution does not leave the tube U_1 .

2. **The interval** $-\sigma_1 \leq \xi^1 \leq \sigma_2$. Here system (2) is written in the new variables

$$\xi^1 = \mu u^1; \quad \xi^i = \mu^2 u^i, \quad i = 2, \dots, k; \quad \eta^1 = \mu^2 v^1; \quad \eta^j = \mu^3 v^j, \quad j = 2, \dots, l; \quad t = \mu^2 \tau; \quad \varepsilon = \mu^3 :$$

$$\dot{u}^1 = (u^1)^2 + v^1 + \mu F^1, \quad \mu \dot{u}^i = d_{\alpha'}^i u^{\alpha'} + b_1^i v^1 + c_0^i (u^1)^2 + \mu F^i,$$

$$\dot{v}^1 = 1 + \mu a_1^1 u^1 + \mu^2 \Gamma^1, \quad \dot{v}^j = a_1^j u^1 + \mu \Gamma^j, \quad (7)$$

$$i = 2, \dots, k, \quad j = 2, \dots, l.$$

(a dot denotes differentiation with respect to τ). The nonautonomous system obtained from system (7) if u^1 is taken in the latter as the independent variable will be denoted by (7'). As in ¹, we write the formal approximations:

$$u^{i,1} = u_0^i(u^1) + \mu u_1^i(u^1), \quad v^{1,1} = v_0^1(u^1) + \mu v_1^1(u^1), \quad v^{j,0} = v_0^j(u^1),$$

$$i = 2, \dots, k, \quad j = 2, \dots, l.$$

We shall call the **tube** U_2 the set of all points of the space $(u^1, u^2, \dots, u^k, v^1, \dots, v^l)$ whose coordinates satisfy the inequalities

$$W(u^2 - u^{2,1}(u^1), \dots, u^k - u^{k,1}(u^1)) \leq \mu M_2^2, \quad |v^1 - v^{1,1}(u^1)| \leq \mu N_2, \quad |v^j - v^{j,0}| \leq P_2, \quad j = 2, \dots, l,$$

where M_2, N_2, P_2 are positive constants independent of ε . The set of points of the tube U_2 singled out by the equation

$$W(u^2 - u^{2,1}(u^1), \dots, u^k - u^{k,2}(u^1)) = \mu M_2^2$$

will be called the **W-wall** and denoted by U_2^W .

Lemma 2. *On the interval $-\sigma_1/\mu \leq u^1 \leq \sigma_2/\mu$, the wall U_2^W of the tube U_2 , for sufficiently large M_2 , is a surface without contact for the system of equations (7'), and all trajectories of system (7) that begin on the wall U_2^W , as τ increases, enter the tube U_2 .*

With the aid of Lemma 2, by the method of successive approximations it is proved that, if the initial point of some solution of system (7') for $u^1 = -\sigma_1/\mu$ is taken in the tube U_2 , then throughout the interval $-\sigma_1/\mu \leq u^1 \leq \sigma_2/\mu$ this solution does not leave the tube U_2 .

3. **The interval** $\sigma_2 \leq \xi^1 \leq p$. Let $\xi^i = \xi_0^i(\xi^1)$ be a solution of the system of equations

$$\frac{d\xi^i}{d\xi^1} = \frac{\Phi^i(\xi, 0)}{\Phi^1(\xi, 0)}, \quad i = 2, \dots, k, \quad (8)$$

defined for $\xi^1 > 0$ and tending to zero as $\xi^1 \rightarrow 0$.

We shall call the **tube** U_3 the set of all points of the space $(\xi^1, \dots, \xi^k, \eta^1, \dots, \eta^l)$ whose coordinates satisfy the inequalities

$$W(\xi^2 - \xi_0^2(\xi^1), \dots, \xi^k - \xi_0^k(\xi^1)) = [R\varepsilon^{4/s}]^2, \quad |\eta^j| \leq L\varepsilon^{2/s},$$

where R and L are positive constants. The set of points of the tube U_3 singled out by the equation

$$W(\xi^2 - \xi_0^2(\xi^1), \dots, \xi^k - \xi_0^k(\xi^1)) = [R\varepsilon^{4/s}]^2$$

will be called the W -wall and denoted by U_3^W .

Lemma 3. *On the interval $\sigma_2 \leq \xi^1 \leq p$, the wall U_3^W of the tube U_3 , for sufficiently large R , is a surface without contact for the system of equations (2), and all trajectories of system (2) that begin on the wall U_3^W , as t increases, enter the tube U_3 .*

With the aid of Lemma 3, by the method of successive approximations it is proved that, if the initial point of some solution of system (3) for $\xi^1 = \sigma_2$ is taken in the tube U_3 , then throughout the interval $\sigma_2 \leq \xi^1 \leq p$ this solution does not leave the tube U_3 .

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