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Abstract

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ON THE CONVERGENCE OF FORMAL SOLUTIONS OF NONLINEAR INTEGRAL EQUATIONS

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One of the basic methods for solving integral equations with analytic nonlinearities was proposed by A. I. Nekrasov ([1]). This method consists in finding solutions in the form of series in integral or fractional powers of a small parameter. To construct these series, the method of undetermined coefficients is applied, and then, to prove their convergence, majorant numerical series are constructed. The principal difficulty, naturally, lies in constructing these majorants. After the works of A. I. Nekrasov, such majorants for various cases were constructed in the works of N. N. Nazarov (for example ([2])), and then in the works of other authors. In particular, a very general case was considered by P. P. Rybin ([3]).

M. A. Krasnosel' skii advanced the hypothesis that, for integral equations of the usual type (those considered in the Lyapunov–Schmidt theory), formal solutions in the form of series are always true solutions for sufficiently small values of the parameters. This hypothesis proved to be correct.

The present paper contains the corresponding theorems for the nonlinear integral equation of P. S. Uryson of general form

$$\varphi(x) = \int_0^1 K[x, y, \varphi(y); \alpha]\, dy. \tag{1}$$

To prove the convergence of formal solutions, it was necessary to generalize the theory of formal power series with numerical coefficients, developed by S. Bochner and W. Martin ([4]), to the case of formal power series with functional coefficients. The proof of the main theorem is based on the use of the ideas of Lyapunov–Schmidt ([5,6]). From the principal result of the paper it follows that, in order to construct all small solutions of equation (1), one must seek, by the method of undetermined coefficients, formal series satisfying the equation: they will be true solutions. Equation (1) can have no other small solutions ([7]).

1. Consider the collection Φ_m of expressions of the form

$$\varphi(M;\alpha) = \varphi(M;\alpha_1,\dots,\alpha_m) = \sum_{k_1+\dots+k_m\geq 1} \varphi_{k_1\dots k_m}(M)\alpha_1^{k_1}\cdots\alpha_m^{k_m}, \qquad (2)$$

where $M = M(x_1,\dots,x_n)$ is a point of the n-dimensional cube D; $\varphi_{k_1\dots k_m}(M)$ are functions defined in D, and $\alpha_1,\dots,\alpha_m$ are complex variables. Expression (2) is called a **formal power series with functional coefficients**, and $\varphi(M;\alpha)$ the **formal sum** of this series. Denote by Φ_m^0 the subset of Φ_m containing formal power series with numerical coefficients. By definition, the series (2) vanishes if all $\varphi_{k_1\dots k_m}(M) \equiv 0$ in D.

The notions of equality of two series $\varphi(M;\alpha)$ and $\psi(M;\alpha)$, of the algebraic sum $a(M)\varphi(M;\alpha) + b(M)\psi(M;\alpha)$ with functional coefficients $a(M)$ and $b(M)$, of the product $\varphi(M;\alpha)\cdot\psi(M;\alpha)$, of the power $[\varphi(M;\alpha)]^k$, and of a double formal series with formal sum $\varphi[M;\psi(M;\alpha)]$, are extended to such series in the usual way.

In what follows it is assumed that all functions $\varphi_{k_1\dots k_m}(M)$ are real and continuous in D.

Integration of formal series is defined as termwise integration of their coefficients. As a result of integration with respect to some of the variables $(x_1,\dots,x_m)$, one obtains a formal power series with functional coefficients depending on the remaining variables, and, in particular, in the case of integration over the whole domain D, a formal power series with numerical coefficients.

The basic propositions of the topology of formal power series with numerical coefficients, considered by S. Bochner and W. Martin, can also be extended to spaces of series with functional coefficients. It seems natural to apply topological methods [9] to the study, as a whole, of the behavior of solutions of integral equations with analytic nonlinearities.

2. Let $K(x,y;z,\alpha)$ be the formal power series

$$K(x,y;z,\alpha) = \sum_{k_0+k_1+\dots+k_m\geqslant 1} K_{k_0k_1\dots k_m}(x,y)z^{k_0}\alpha_1^{k_1}\cdots\alpha_m^{k_m},$$

where $z \in \Phi_m$.

Introduce the notation: $K_{100\dots0}(x,y) = K(x,y)$, $K_{010\dots0}(x,y) = K_1(x,y), \dots$

$$\dots,\ K_{000\dots01}(x,y) = K_m(x,y), \qquad \sum_{k_0+k_1+\dots+k_m\geqslant 2} K_{k_0k_1\dots k_m}(x,y)z^{k_0}\alpha_1^{k_1}\cdots\alpha_m^{k_m} =$$

$$= \Gamma(x,y;z,\alpha)$$

and consider the nonlinear integral equation

$$\varphi(x;\alpha) = \int_0^1 K(x,y)\varphi(y;\alpha)\,dy + \sum_{i=1}^{m} \alpha_i \int_0^1 K_i(x,y)\,dy+$$

$$+\int_0^1 \Gamma[x,y;\varphi(y;\alpha),\alpha]\,dy. \tag{3}$$

By a formal solution of this equation we shall mean such a formal power series $\varphi(x;\alpha)$ with functional coefficients that substitution of it into (3) leads to the coincidence of the formal power series standing on the right- and left-hand sides of (3), i.e., to the identity of the coefficients of like terms $\alpha_1^{k_1} \cdots \alpha_m^{k_m}$.

Theorem 1. *If* 1 *is not an eigenvalue of the kernel* $K(x,y)$*, then equation (3) has a unique solution in the class of formal power series.*

In the case where 1 is an eigenvalue of the kernel $K(x,y)$ of rank r, let us consider the auxiliary equation

$$\psi(x;\alpha,\beta) = \int_0^1 L(x,y)\psi(y;\alpha,\beta)\,dy + \sum_{i=1}^{m} \alpha_i \int_0^1 K_i(x,y)\,dy+$$

$$+\int_0^1 \Gamma\left[x,y;\psi(y;\alpha,\beta) + \sum_{j=1}^{r} \beta_j\omega_j(y),\alpha\right] dy, \tag{4}$$

in which $\{\omega_j(x)\}$ and $\{\omega_j^*(x)\}$, respectively, are a system of r linearly independent eigenfunctions of the kernel $K(x,y)$ and of the transposed kernel

$K(y,x)$, corresponding to the eigenvalue 1; $\beta_1,\dots,\beta_r$ are complex numerical parameters, and, finally,

$$L(x,y) = K(x,y) - \sum_{j=1}^{r} \omega_j(y)\omega_j^*(x).$$

The kernel $L(x,y)$ does not have 1 as an eigenvalue and, by Theorem 1, equation (4) has a unique formal solution of the form

$$\psi(x;\alpha,\beta) = \sum_{(k,l)} \psi_{(k,l)}(x)\alpha_1^{k_1}\dots\alpha_m^{k_m}\beta_1^{l_1}\dots\beta_r^{l_r}, \tag{5}$$

where (k,l) denotes the totality of $m{+}r$ nonnegative integers $(k_1,\dots,k_m,l_1,\dots,l_r)$, for which

$$\sum_{i=1}^{m} k_i + \sum_{j=1}^{r} l_j \geq 1.$$

Subjecting solution (5) to the conditions

$$\int_0^1 \psi(x;\alpha,\beta)\omega_j(x)\,dx = 0 \qquad (j = 1,\dots,r),$$

we arrive at a system of formal equations of connection of the form

$$\sum_{(k,l)} a^{(j)}_{(k,l)} \alpha_1^{k_1} \dots \alpha_m^{k_m} \beta_1^{l_1} \dots \beta_r^{l_r} = 0 \qquad (j = 1,\dots,r), \tag{6}$$

in which

$$a^{(j)}_{(k,l)} = \int_0^1 \psi_{(k,l)}(x)\omega_j(x)\,dx.$$

We shall call system (6) the **formal system of branching equations** for the integral equation (3).

Theorem 2. *If 1 is an eigenvalue of the kernel $K(x,y)$ of rank r, then the number of formal solutions of equation (3) is equal to the number of formal solutions of system (6), regarded as a system defining formal implicit functions $\beta_1,\dots,\beta_r$ of the variables $\alpha_1,\dots,\alpha_m$.*

Let us note that the coefficients $a^{(j)}_{(k,l)}$ of system (6) can be effectively computed from equation (3).

3. Restricting ourselves to the particular case $m = 1$, $r = 1$, let us now turn to the main fact. Let $\varphi(x;\alpha)$ be a formal solution of equation (3). Represent $\varphi(x;\alpha)$ in the form $\psi(x;\alpha) + a(\alpha)\omega(x)$, where $\psi(x,\alpha)$ is a formal sum of a series, orthogonal (with respect to the coefficients) to the function $\omega(x)$, and $a(\alpha) \in \Phi_1^0$. It can be shown that $a(\alpha)$ satisfies an equation of the form $P(\alpha,\beta) = 0$ (where $P(\alpha,\beta)$ is a normalized irreducible pseudopolynomial), which has a unique solution $\beta = a(\alpha)$, representable for small values of the parameter α by a convergent power series in α. It further turns out that the formal sum $\psi(x;\alpha)$ satisfies a nonlinear integral equation of the form (4), for which 1 is not an eigenvalue of the kernel $L(x,y)$, which, by Theorem 1, has a unique solution. This solution, for small values of the parameters α and $a(\alpha)$, is at the same time the true solution of this equation. Thus, under simplifying assumptions, the following holds:

Theorem 3. *If for all $K_{k_0 k_1}(x,y)$ the condition*

$$|K_{k_0 k_1}(x,y)| \leq A < \infty \qquad (A = \text{const})$$

is satisfied, and 1 is a simple eigenvalue of the kernel $K(x,y)$, then every formal solution

$$\varphi(x;\alpha) = \sum_{k=1}^{\infty} \varphi_k(x)\alpha^k$$

is a true solution of this equation for sufficiently small α.

When carrying out the method of undetermined coefficients for finding formal solutions of equation (3), there also appear series arranged in fractional powers of the small parameter α, of the form

$$\varphi(x;\alpha) = \sum_{k=1}^{\infty} \varphi_k(x)\alpha^{k/s} \quad (s \text{ is an integer, } s \geqslant 1). \tag{7}$$

This case is covered by the considerations indicated above, since it is sufficient in the series (7) and in equation (3) to put $\alpha = \alpha {a'}^s$.

Let us note, finally, that the application of a theorem of N. P. Erugin [8] makes it possible to extend Theorem 3 to the case when $r > 1$.

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Note: Figure translations are in progress. See original paper for figures.