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Abstract

Full Text

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ON A PLANE LINEAR PROBLEM OF GENERALIZED HYDRODYNAMICS*

(Presented by Academician N. N. Bogolyubov on 19 VI 1957)

1. In the paper ([1]) it was shown that the Boltzmann kinetic equation is the first approximation in the solution of the "Bogolyubov chain" with respect to the quantity n/n_0 (n is the number density of particles, $n_0 = 1/r_0^3$, where r_0 is the constant of short-range action), while no restrictions are imposed on the ratio of the relaxation time Δt_p to the time interval Δt characteristic for the process under consideration. Consequently, it is legitimate to pose the question of such a method for solving the Boltzmann equation which would not impose restrictions on $\Delta t_p/\Delta t$. From this point of view, let us consider the method of "moments." To this end we shall obtain equations for the moments of the single-particle distribution function, without using Maxwell' s transport equations or the conservation equations of a continuous medium, as was done in ([2]), but proceeding only from the Boltzmann equation and the known normalization conditions. We write the Boltzmann equation and the normalization conditions in terms of the dimensionless distribution function g and the dimensionless relative velocity of molecules $\vec{\xi}$:

$$\frac{dg}{dt}+c_e\xi_i\frac{\partial g}{\partial x_i}+\frac{1}{c_e}\frac{\partial g}{\partial \xi_i}\left(X_i-\frac{du_i}{dt}\right)-\xi_j\frac{\partial g}{\partial \xi_j}\left(\frac{d\ln c_e}{dt}+c_e\xi_i\frac{\partial \ln c_e}{\partial x_i}\right)-\xi_i\frac{\partial g}{\partial \xi_j}\frac{\partial u_j}{\partial x_i}+g\left[\frac{d\ln(n/c_e^3)}{dt}+c_e\xi_i\frac{\partial \ln(n/c_e^3)}{\partial x_i}\right]= \quad (1)$$

$$\int g\,d\vec{\xi}=1;\qquad \int \xi_i g\,d\vec{\xi}=0;\qquad \int \xi^2 g\,d\vec{\xi}=3;\qquad P_{ij}=\rho c_e^2\int \xi_i\xi_j g\,d\vec{\xi};$$

$$q_i=\frac{\rho c_e^3}{2}\int \xi_i\xi^2 g\,d\vec{\xi};\qquad S_{ijk}=\rho c_e^3\int \xi_i\xi_j\xi_k g\,d\vec{\xi}\quad \text{etc.},$$

where

$$g(t,\mathbf{r},\vec{\xi})=f\,\frac{c_e^3(t,\mathbf{r})}{n(t,\mathbf{r})};\qquad \vec{\xi}=\frac{\mathbf{c}}{c_e};\qquad \xi^2=\xi_1^2+\xi_2^2+\xi_3^2;$$

$$c_e = \sqrt{\frac{2}{3}\, e(t, \mathbf{r})}; \qquad e(t, \mathbf{r}) = \frac{1}{\rho} \int \frac{mc^2}{2} f \, d\mathbf{c}; \qquad \frac{d}{dt} = \frac{\partial}{\partial t} + u_i \frac{\partial}{\partial x_i};$$

$$J(gg_1) = \int c_e |\vec{\xi}_1 - \vec{\xi}| \{g' g_1' - g g_1\} \, b \, db \, d\varphi \, d\vec{\xi}_1.$$

Substitute in (1) the expansion of g in generalized Hermite polynomials $H^{(r)}(\vec{\xi})$ ([3])

$$g = g_0 \sum_{r=0}^{\infty} \sum_{i_1 \dots i_r} \frac{1}{r!} \alpha^{(r)}_{i_1 \dots i_r} H^{(r)}_{i_1 \dots i_r}, \tag{2}$$

* Reported at the All-Union Acoustics Conference on 25 VI 1957.

where $g_0 = (2\pi)^{-3/2} \exp(-\xi^2/2)$; $\alpha^{(r)}_{i_1 \dots i_r}$ are unknown functions of $t, \mathbf{r}$. (By virtue of the orthonormality of the functions $H^{(r)}$, the $\alpha^{(r)}$ are related to the moments of the function g.) We obtain*:

$$\begin{aligned}
&\frac{d\alpha^{(s)}_{j_1 \cdots j_s}}{dt} + c_e \frac{\partial}{\partial x_\gamma} \left\{ \alpha^{(s+1)}_{\gamma j_1 \dots j_s} + \sum_{\hat{P}_{j_1 \cdots j_s}} \delta_{\gamma j_1} \alpha^{(s-1)}_{j_2 \cdots j_s} \right\} \\
&\quad + \frac{1}{c_e} \left(\frac{du_\gamma}{dt} - X_\gamma \right) \sum_{\hat{P}_{j_1 \cdots j_s}} \delta_{\gamma j_1} \alpha^{(s-1)}_{j_2 \cdots j_s} \\
&\quad + \frac{d \ln(n c_e^s)}{dt} \alpha^{(s)}_{j_1 \cdots j_s} + 2 \frac{d \ln c_e}{dt} \sum_{\hat{P}_{j_1 \cdots j_s}} \delta_{j_1 j_2} \alpha^{(s-2)}_{j_3 \cdots j_s} \\
&\quad + c_e \frac{\partial \ln(n c_e^{s+1})}{\partial x_\gamma} \left\{ \alpha^{(s+1)}_{\gamma j_1 \dots j_s} + \sum_{\hat{P}_{j_1 \cdots j_s}} \delta_{\gamma j_1} \alpha^{(s-1)}_{j_2 \cdots j_s} \right\} \\
&\quad + 2 \frac{\partial c_e}{\partial x_\gamma} \left\{ \sum_{\hat{P}_{j_1 \cdots j_s}} \delta_{j_1 j_2} \alpha^{(s-1)}_{\gamma j_3 \dots j_s} + \sum_{\hat{P}_{j_1 \cdots j_s}} \delta_{j_1 j_2} \sum_{\hat{P}_{j_3 \cdots j_s}} \delta_{\gamma j_3} \alpha^{(s-3)}_{j_4 \cdots j_s} \right\} \\
&\quad + \frac{\partial u_\gamma}{\partial x_\gamma} \alpha^{(s)}_{j_1 \cdots j_s} + \frac{\partial u_\gamma}{\partial x_\nu} \left\{ \sum_{\hat{P}_{j_1 \cdots j_s}} \delta_{\gamma j_1} \alpha^{(s)}_{\nu j_2 \dots j_s} + \sum_{\hat{P}_{j_1 \cdots j_s}} \delta_{\gamma j_1} \sum_{\hat{P}_{j_2 \cdots j_s}} \delta_{\gamma j_2} \alpha^{(s-2)}_{j_3 \cdots j_s} \right\} = J^{(s)}_{j_1 \cdots j_s},
\end{aligned} \tag{3}$$

where

$$J^{(s)}_{j_1 \cdots j_s} = n \int H^{(s)}_{j_1 \cdots j_s} J(gg_1) \, d\vec{\xi}.$$

Putting $s = 0, 1, 2, \ldots$ and taking into account the relation of $\alpha^{(s)}$ to the moments, we obtain equations for the moments:

$$\frac{dn}{dt} + \frac{\partial n u_\gamma}{\partial x_\gamma} = 0, \qquad \frac{\partial u_{j_1}}{\partial t} + u_\gamma \frac{\partial u_{j_1}}{\partial x_\gamma} = X_{j_1} - \frac{1}{\rho}\frac{\partial P_{\gamma j_1}}{\partial x_\gamma};$$

$$\frac{\partial P_{j_1 j_2}}{\partial t} + u_\gamma \frac{\partial P_{j_1 j_2}}{\partial x_\gamma} + \frac{\partial S_{\gamma j_1 j_2}}{\partial x_\gamma} + \frac{\partial u_\gamma}{\partial x_\gamma} P_{j_1 j_2} + \frac{\partial u_{j_1}}{\partial x_\gamma} P_{\gamma j_2} + \frac{\partial u_{j_2}}{\partial x_\gamma} P_{\gamma j_1} = \rho c_e^2 J^{(2)}_{j_1 j_2}, \quad (4)$$

$$\frac{\partial S_{j_1 j_2 j_3}}{\partial t} + u_\gamma \frac{\partial S_{j_1 j_2 j_3}}{\partial x_\gamma} + \frac{\partial u_\gamma}{\partial x_\gamma} S_{j_1 j_2 j_3} + \sum_{\hat{P}_{j_1 j_2 j_3}} \frac{\partial u_{j_1}}{\partial x_\gamma} S_{\gamma j_2 j_3} - \frac{1}{\rho} \sum_{\hat{P}_{j_1 j_2 j_3}} P_{j_1 j_2} \frac{\partial P_{\gamma j_3}}{\partial x_\gamma}$$
$$+ \frac{\partial}{\partial x_\gamma} \sum_{\hat{P}_{j_1 j_2 j_3}} \left\{ c_e^2 P_{\gamma j_1} \delta_{j_2 j_3} + c_e^2 P_{j_2 j_3} \delta_{\gamma j_1} - \rho c_e^4 \delta_{\gamma j_1} \delta_{j_2 j_3} \right\} + \frac{\partial}{\partial x_\gamma} \{ \rho c_e^4 \alpha^{(4)}_{\gamma j_1 j_2 j_3} \} = \rho c_e^3 J^{(3)}_{j_1 j_2 j_3}.$$

Restriction to a finite number of moments imposes only the requirement $\alpha^{(r)}_{j_1 \cdots j_r}\big|_{r \geq 2} \ll 1$. It follows from the derivation that the method of moments does not require any restriction on the magnitude of $\Delta t_p / \Delta t$. In this connection we shall call the system (4) the **equations of generalized hydrodynamics, valid for describing rapid processes**.

* Here and below, $\hat{P}_{j_1 \cdots j_s}$ denotes summation over the terms, distinct from one another, obtained as a result of permutation of the indices.

** The calculation of $J^{(2)}_{j_1 j_2}$ and $J^{(3)}_{j_1 j_2 j_3}$ from (4) was carried out in (2). In (5) we have restricted ourselves to terms linear in $J^{(2)}_{j_1 j_2}$ and $J^{(3)}_{j_1 j_2 j_3}$ with respect to $\alpha^{(s)}$. $B^{(2)}_1$ depends on T, m, and the law of interaction between the particles.

2. To solve the plane linear problem of the propagation of small perturbations with allowance for the processes of transfer of momentum and energy, six quantities are necessary: n, u_1, p_{11}, $p_{22} + p_{33}$, S_{111}, $S_{122} + S_{123}$, or n, u_1, p_{11}, $p = \frac{1}{3}(p_{11} + p_{22} + p_{33})$, S_{111}, $S_1 = S_{111} + S_{122} + S_{133}$. For small deviations from the equilibrium stationary state, in the case considered, from (4) we obtain

$$\frac{\partial \beta^{(0)}}{\partial t} + (c_\varepsilon)_0 \frac{\partial \beta^{(1)}_1}{\partial x_1} = 0; \qquad \frac{\partial \beta^{(1)}_1}{\partial t} + (c_\varepsilon)_0 \frac{\partial \beta^{(2)}_{11}}{\partial x_1} = 0;$$

$$\frac{\partial \beta^{(2)}_{11}}{\partial t} + 3(c_\varepsilon)_0 \frac{\partial \beta^{(1)}_1}{\partial x_1} + (c_\varepsilon)_0 \frac{\partial \beta^{(3)}_{111}}{\partial x_1} = 6(\eta)_0 B^{(2)}_1 \left(\beta^{(2)} - \beta^{(2)}_{11} \right);$$

$$\frac{\partial \beta^{(2)}}{\partial t} + \frac{5}{3}(c_\varepsilon)_0 \frac{\partial \beta_1^{(1)}}{\partial x_1} + \frac{1}{3}(c_\varepsilon)_0 \frac{\partial \beta_1^{(3)}}{\partial x_1} = 0; \tag{5}$$

$$\frac{\partial \beta_{111}^{(3)}}{\partial t} - 3(c_\varepsilon)_0 \frac{\partial \beta^{(0)}}{\partial x_1} + 3(c_\varepsilon)_0 \frac{\partial \beta_{11}^{(2)}}{\partial x_1} = 3(\eta)_0 B_1^{(2)} \left(\beta_1^{(3)} - 3\beta_{111}^{(3)}\right);$$

$$\frac{\partial \beta_1^{(3)}}{\partial t} - 5(c_\varepsilon)_0 \frac{\partial \beta^{(0)}}{\partial x_1} + 2(c_\varepsilon)_0 \frac{\partial \beta_{11}^{(2)}}{\partial x_1} + 3(c_\varepsilon)_0 \frac{\partial \beta^{(2)}}{\partial x_1} = -4(\eta)_0 B_1^{(2)} \beta_1^{(3)},$$

Table 1

Dimensionless velocities V/V_0 and attenuations per mean free path $\varkappa l$ as functions of $\varepsilon = \nu\mu/p$ $\left(V_0 = V_1\big|_{\varepsilon\to 0};\ l = \dfrac{8}{15\sqrt{2\pi}} \dfrac{c_\varepsilon}{nB_1^{(2)}}\right)$

ε	$\left(\frac{V}{V_0}\right)_1$	$\varkappa_1 l$	$\left(\frac{V}{V_0}\right)_2$	$\varkappa_2 l$	$\left(\frac{V}{V_0}\right)_3$	$\varkappa_3 l$
0.00265	1.00030	0.00019	0.17020	0.09349	–	1.2
0.00531	1.00119	0.00077	0.23659	0.12989	0.7	1.2
0.0106	1.00475	0.00305	0.32335	0.17721	0.7	1.2
0.0159	1.01061	0.00677	0.38287	0.20922	0.7	1.2
0.0212	1.01870	0.01183	0.42756	0.23272	0.7	1.2
0.0265	1.02892	0.01811	0.46244	0.25046	0.7	1.2
0.0318	1.04119	0.02543	0.49015	0.26396	0.7	1.2
0.0371	1.05542	0.03363	0.51233	0.27448	0.7	1.2
0.0424	1.07150	0.04250	0.53008	0.28184	0.7	1.2
0.0531	1.10852	0.06140	0.55549	0.29171	0.7	1.2
0.0796	1.21534	0.10679	0.58433	0.30189	0.7	1.2
0.1061	1.31403	0.14199	0.59336	0.30930	0.7	1.2
0.1326	1.39325	0.16875	0.59666	0.31859	0.7	1.2
0.3	1.64213	0.25305	0.61528	0.3359	0.8	1.1
0.5	1.73712	0.28213	0.58251	0.45885	0.8	1.0
0.7	1.76981	0.29167	0.57924	0.46574	0.8	1.0
1.0	1.78579	0.2997	0.57699	0.47490	0.8	1.0

where

$$\beta^{(0)} = \Delta n/(n)_0, \qquad \beta_1^{(1)} = \Delta u_1/(c_\varepsilon)_0;$$

$$\beta_{11}^{(2)} = \Delta p_{11}/(\rho c_\varepsilon^2)_0, \qquad \beta^{(2)} = \Delta p/(\rho c_\varepsilon^2)_0;$$

$$\beta_{111}^{(3)} = \Delta S_{111}/(\rho c_\varepsilon^3)_0, \qquad \beta_1^{(3)} = \Delta S_1(\rho c_\varepsilon^3)_0.$$

To equation (5), in the case of a prescribed frequency, there corresponds the dispersion equation

$$\{[1-\tfrac{2}{3}\cdot 19(\pi\varepsilon)^2]+i[\tfrac{57}{9}(\pi\varepsilon)-8(\pi\varepsilon)^3]\}u^6+\{[-\tfrac{5}{3}+\tfrac{172}{3}(\pi\varepsilon)^2]+i[-\tfrac{164}{9}(\pi\varepsilon)+56(\pi\varepsilon)^3]\}u^4$$
$$+\{[-\tfrac{98}{3}(\pi\varepsilon)^2]+i[5(\pi\varepsilon)-72(\pi\varepsilon)^3]\}u^2+i[24(\pi\varepsilon)^3]=0, \qquad (6)$$

where

$$u=\omega/k(c_\varepsilon)_0; \qquad (c_\varepsilon)_0=\sqrt{kT/m};$$

$\varepsilon=\nu/6nB_1^{(2)}=\nu\mu/p$; ν is the wave frequency; p is the gas pressure;

μ is the first approximation of the Chapman viscosity coefficient.

The results of the numerical solution of (6) are given in Table 1.

If $\varepsilon=\nu\dfrac{\mu}{p}\ll 1$, then (6), with accuracy up to ε^2, gives

$$V_1=V_0\left[1+4.3\pi^2\left(\frac{\nu\mu}{p}\right)^2\right]; \qquad \varkappa_1=\frac{42\pi^2}{5\sqrt{15}}\frac{\nu^2\mu}{p}\sqrt{\frac{m}{kT}}=\frac{7}{6}\frac{\omega^2\mu}{\rho V_0^3}; \qquad (7)$$

$$V_2=\sqrt{\frac{6\pi\nu\mu kT}{pm}}\left[1-1.3\pi\left(\frac{\nu\mu}{p}\right)\right]; \qquad \varkappa_2=\sqrt{\frac{2\pi\nu pm}{3\mu kT}}\left[1-1.3\pi\left(\frac{\nu\mu}{p}\right)\right]. \quad (8)$$

From (7) and (8), neglecting the second terms in the brackets, we obtain the well-known expressions for the adiabatic speed of sound and for the velocity and attenuation of "thermal waves" [4]:

$$V_0=\sqrt{\frac{5}{3}\frac{kT}{m}}; \qquad (V_2)_0=\sqrt{2\chi\omega}; \qquad (\varkappa_2)_0=\sqrt{\frac{\omega}{2\chi}}; \qquad \chi=\frac{3}{2}\frac{\mu}{\rho}=\frac{\lambda}{\rho c_p},$$

where χ is the thermal diffusivity. The attenuation of sound x_1 coincides with the known expression [4] when the first viscosity and thermal conductivity are taken into account. From consideration of the numerical solution of (6) and (7), (8), it follows: the first solution—the "acoustic branch" —gives the translational dispersion of sound [5–8] and describes the boundary of propagation of ultrasound in monatomic gases as a function of various parameters [9]; the second solution gives the same complete description of "thermal waves" ; the third solution, apparently, is practically not realized because of the large values of xl_3. A comparison of the first solution with experiment is given in Fig. 1 in the notation of [7]. For brevity of presentation, the graph is given only for the

Fig. 1

Figure 1: Fig. 1

dimensionless velocity. In [7] there is a quantitative discrepancy between experiment and the theories of Navier–Stokes, Burnett, and the "super" -Burnett approximation in the region $r \lesssim 1$ or $\varepsilon \gtrsim 0.16$ $(r = 1/2\pi\varepsilon)$. In fact, the discrepancy is qualitative, since the author uses dispersion equations containing terms of higher order in $1/r$ (or in ε) in comparison with the original hydrodynamic equations. In the theory being developed, this incorrectness is absent, and for the speed of sound the theory agrees with experiment over the entire interval of values of ε (or r). The conclusions of the present work are valid for monatomic gases in the Boltzmann approximation.

Fig. 1

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Note: Figure translations are in progress. See original paper for figures.