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1. The case of outflow with subcritical velocity.

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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

HYDROMECHANICS

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OUTFLOW OF GAS FROM A VESSEL WITH WALLS ENCLOSING A SMALL ANGLE $2\theta_0$

(Presented by Academician L. I. Sedov on 14 VI 1958)

1. The case of outflow with subcritical velocity.
In Fig. 1, Ox is the trace of the plane of symmetry; BB', AA' are the walls of the vessel; $B'B''$, $A'A''$ are the jet. φ and ψ are, respectively, the velocity potential and the stream function; $\psi = 0$ on Ox, while in the region of outflow ψ varies from $-Q/2$ to $Q/2$, where $\rho_0 Q$ is the discharge per 1 sec.

The problem posed can be solved analogously to Chaplygin' s problem ([1]).

For the stream function ψ and the velocity potential φ, the following formulas hold:

$$\frac{\pi}{Q}\psi = -k\theta - \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\tau}{\tau_0}\right)^{kn} \frac{Y_{kn}}{Y_{kn,0}} x_{kn} \delta \sin 2kn\theta,$$

$$-\frac{\pi}{Q}\varphi = c + \frac{1}{2} \int \frac{d\tau}{\tau(1-\tau)^{\beta}} - \frac{1}{(1-\tau)^{\beta}} + \frac{1}{(1-\tau)^{\beta}} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{\tau}{\tau_0}\right)^{kn} \frac{y_{kn}}{y_{kn,0}} x_{kn} \cos 2kn\theta, \tag{1}$$

where $k = \pi/2\theta_0$; θ is the angle of inclination of the velocity; $\tau = w^2/w_{\max}^2$; w is the modulus of the flow velocity; $w_{\max}$ is the outflow velocity into vacuum.

Fig. 1

Using these formulas, one can find the equation of the jet boundary on which $\tau = \tau_0$. We have

$$\frac{\pi}{Q} w_0 (1-\tau)^{\beta} y = -k \sin\theta - \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{2kn}{4k^2h^2 - 1} \left\{ (x_{kn} + 2kn) \sin 2kn\theta \cos\theta - (1 + 2kn x_{kn}) \cos 2kn\theta \sin\theta \right\}, \tag{2}$$

where $x_{kn}(\tau)$ is an auxiliary function introduced by Chaplygin. At the outlet $\theta = \theta_0$, $y = -a$. Consequently:

$$w_0(1-\tau)^\beta \frac{\pi}{Q} a = k \sin\theta_0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{2kn}{4k^2n^2-1}(1+2knx_{kn}) \sin\theta_0. \tag{3}$$

Taking into account that $Q = 2b(1-\tau_0)^\beta w_0$, we obtain

$$\frac{\pi}{2}\frac{a}{b} = k \sin\theta_0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{2kn}{4k^2n^2-1}(1+2knx_{kn}) \sin\theta_0. \tag{4}$$

Let us consider the sum in (4):

$$I = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \frac{2kn}{4k^2n^2-1}\left(1+\frac{1}{4k^2n^2}+\frac{1}{16k^4n^4}+\cdots\right)(1+2knx_{kn}). \tag{5}$$

Guderley and Light-hill showed that, for Mach number $\mathrm{M} < 1$, there exists an asymptotic expansion of the form

$$x_{kn} \sim x^{(0)}(\tau) + \frac{x^{(1)}(\tau)}{kn} + \frac{x^{(2)}(\tau)}{k^2n^2} + \frac{x^{(3)}(\tau)}{k^3n^3} + \cdots \tag{6}$$

Applying Chaplygin' s method, F. I. Frankl found the recurrence formulas:

$$\begin{aligned}
x^{(0)} &= \sqrt{1-2\beta s},\\
x^{(1)} &= -\frac{1}{2x^{(0)}}\,[s(1+s)x^{(0)\prime} + \beta s x^{(0)}],\\
x^{(2)} &= -\frac{1}{2x^{(0)}}\,[s(1+s)x^{(1)\prime} + \beta s x^{(1)} + x^{(1)^2}],\\
x^{(3)} &= -\frac{1}{2x^{(0)}}\,[s(1+s)x^{(2)\prime} + \beta s x^{(2)} + 2x^{(1)}x^{(2)}],\\
&\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{7}$$

where $\beta = \dfrac{1}{\kappa-1}$, $s = \dfrac{\tau}{1-\tau} = \dfrac{1}{2\beta}\mathrm{M}^2$. Substituting s and β, we obtain expressions for $x^{(i)}$ in terms of M:

$$\begin{aligned}
x^{(0)} &= \sqrt{1-\mathrm{M}^2},\\
x^{(1)} &= \frac{\kappa+1}{8}\frac{\mathrm{M}^4}{1-\mathrm{M}^2},\\
x^{(2)} &= -\frac{\kappa+1}{32}\frac{\mathrm{M}^4}{(1-\mathrm{M}^2)^{5/2}}\left[4-(3-2\kappa)\mathrm{M}^2-\frac{3\kappa-1}{4}\mathrm{M}^4\right],
\end{aligned} \tag{8}$$

$$x^{(3)} = -\frac{[2+(\kappa-1)\mathrm{M}^2]\,\mathrm{M}^4}{64(1-\mathrm{M}^2)^4}\Big\{[2+(\kappa-1)\mathrm{M}^2][\kappa+6(1-\kappa)\mathrm{M}^2+(5\kappa-1)\mathrm{M}^4$$
$$-\kappa\mathrm{M}^6](1-\mathrm{M}^2)^{1/2} + \frac{2(\kappa-1)(1-\mathrm{M}^2)^{7/2}}{\mathrm{M}^2}[2+5\mathrm{M}^2(1-\mathrm{M}^2)]$$
$$-(\kappa+1)\mathrm{M}^2\left[4-(3-2\kappa)\mathrm{M}^2-\frac{3\kappa-1}{4}\mathrm{M}^4\right]\Big\}.$$

Substituting the value of x_{kn} into equation (5), we obtain

$$I = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\left(\frac{1}{2kn}+\frac{1}{8k^2n^2}+\frac{1}{32k^4n^4}+\cdots\right)\left(1+2knx^{(0)}+2x^{(1)}+\right.$$
$$\left.+\frac{2x^{(3)}}{kn}+\frac{2x^{(3)}}{k^2n^2}+\cdots\right)$$
$$= 0.693x^{(0)} + 0.822\,\frac{1/2+x^{(1)}}{k} + 0.901\,\frac{x^{(2)}+\frac{1}{4}x^{(0)}}{k^2} + 0.951\,\frac{1/8+\frac{1}{4}x^{(2)}+x^{(3)}}{k^3} + \cdots.$$

Substituting this sum into (4), we obtain

$$\frac{\pi}{2}\frac{a}{b} \sim \frac{\pi}{2}\frac{\sin\theta_0}{\theta_0} + \sin\theta_0\left[0.693x^{(0)} + 0.822\,\frac{1/2+x^{(1)}}{\pi}\,2\theta_0 + 0.901\,\frac{x^{(2)}+\frac{1}{4}x^{(0)}}{\pi^2}\,4\theta_0^2 + 0.951\,\frac{1/8+\frac{1}{4}x^{(2)}+x^{(3)}}{\pi^3}\,8\theta_0^3 +\right.$$

or

$$\frac{\pi}{2}\frac{a}{b} \sim \frac{\pi}{2} + 0.693\chi^{(0)}\theta_0 - \left[\frac{\pi}{2} - 1.644\,\frac{1/2+\chi^{(1)}}{\pi}\right]\theta_0^2 +$$
$$+\left[\frac{3.604}{\pi^2}\left(\chi^{(2)}+\frac{1}{4}\chi^{(0)}\right) - \frac{0.693}{6}\chi^{(0)}\right]\theta_0^3 +$$
$$+\left[\frac{\pi}{240} + \frac{7.608}{\pi^3}\left(\frac{1}{8}+\frac{1}{4}\chi^{(2)}+\chi^{(3)}\right) - \frac{0.822}{3\pi}\left(\frac{1}{2}+\chi^{(1)}\right)\right]\theta_0^4 + \cdots$$

Finally we have

$$\frac{b}{a} \sim 1 - 0.441\chi^{(0)}\theta_0 + \left(0.194{\chi^{(0)}}^2 + 0.332\chi^{(1)}\right)\theta_0^2 -$$
$$-\left(0.292\chi^{(0)}\chi^{(1)} - 0.232\chi^{(2)} + 0.015\chi^{(0)} + 0.085{\chi^{(0)}}^3\right)\theta_0^3 +$$

Fig. 2

Figure 2: Fig. 2

$$+\left[0.017+0.055\chi^{(1)}-0.003\chi^{(3)}-0.882\chi^{(0)}\left(0.232\chi^{(2)}-0.015\chi^{(0)}\right)+\right.$$

$$\left.+0.11{\chi^{(1)}}^2+0.128{\chi^{(0)}}^2\chi^{(1)}\right]\theta_0^4+\cdots \tag{9}$$

In order to determine b/a for a diatomic ideal gas, we substitute its value 1.4 for χ in formulas (8) and (9).

The same formulas with $\chi = 2$ apply to the flow of water in a shallow channel ([2]), and also to an ultrarelativistic gas and to a photon gas ([3]). In Table 1 and in Fig. 2 the values of b/a are given.

2. Case of outflow of a liquid with critical velocity

In this case Chaplygin' s auxiliary function is given by the asymptotic formula of F. I. Frankl ([4])

$$x_{kn} \simeq x^*_{kn} \sim \frac{C_0}{n^{1/3}k^{1/3}} + \frac{C_1}{kn} + \frac{C_2}{n^{5/3}k^{5/3}} + \cdots \tag{10}$$

Fig. 2

Table 1

M	$\chi = 1.4$, $\theta_0 = 5^\circ$	$\chi = 1.4$, $\theta_0 = 10^\circ$	$\chi = 1.4$, $\theta_0 = 15^\circ$	$\chi = 2$, $\theta_0 = 5^\circ$	$\chi = 2$, $\theta_0 = 10^\circ$	$\chi = 2$, $\theta_0 = 15^\circ$
0.1	0.9640	0.9293	0.8964	0.964	0.9293	0.8964
0.2	0.9642	0.9303	0.898	0.9642	0.9303	0.898
0.3	0.9653	0.9316	0.9016	0.9652	0.9316	0.9016
0.4	0.9664	0.9343	0.9043	0.9665	0.9342	0.9043
0.5	0.9682	0.9377	0.909	0.9682	0.9378	0.9091
0.6	0.9706	0.9424	0.9155	0.9706	0.9427	0.9158
0.7	0.9738	0.9487	0.9222	0.9741	0.949	0.925
0.8	0.9785	0.9575	0.9366	0.9787	0.9581	0.9376

Substituting x^*_{kn} into formula (4), we obtain

$$\frac{\pi}{a}\frac{a}{b} \sim k\sin\theta_0 + \sin\theta_0 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\frac{1}{2kn}\left(1+\frac{1}{4k^2n^2}+\frac{1}{16n^4k^4}+\cdots\right)\times$$

$$\times \left[1 + 2kn\left(\frac{C_0}{n^{1/3}k^{1/3}} + \frac{1/2 + C_1}{nk} + \frac{C_2}{n^{5/3}k^{5/3}} + \cdots\right)\right] =$$

$$= k\sin\theta_0 + \sin\theta_0 \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\left(\frac{C_0}{n^{1/3}k^{1/3}} + \frac{1/2 + C_1}{nk} + \frac{C_2}{n^{5/3}k^{5/3}} + \cdots\right).$$

After summation we obtain

$$\frac{\pi}{a}\frac{a}{b} = k\sin\theta_0 + \sin\theta_0\left[0.6C_0\theta_0^{2/3} + 0.523(0.5 + C_1)\theta_0 + 0.649C_2\theta_0^{5/3}\right]. \quad (11)$$

By means of transformations analogous to those carried out in the preceding paragraph, we obtain

$$\frac{a}{b} \sim 1 + \frac{1.2}{\pi}C_0\theta_0^{4/3} + \left[\frac{1.04(1.5 + C_1)}{\pi} - 0.1666\right]\theta_0^2 + \frac{1.29}{\pi}C_2\theta_0^{8/3}. \quad (12)$$

In the case of an ideal diatomic gas ($\varkappa = 1.4$), $C_0 = 0.7745$, $C_1 = -0.3901$, $C_2 = 0.2053$ ([4]). Substituting these values of C_i into formula (12), we obtain

$$\frac{a}{b} \sim 1 + 0.295\theta_0^{4/3} - 0.13\theta_0^2 + 0.084\theta_0^{8/3} + \cdots, \quad (13)$$

whence

$$\frac{b}{a} \sim 1 - 0.295\theta_0^{4/3} + 0.13\theta_0^2 - 0.171\theta_0^{8/3} + \cdots \quad (14)$$

In the case of the outflow of water and a photon gas ($\varkappa = 2$) we have $C_0 = 0.834$, $C_1 = 0.450$, so that, according to (12):

$$\frac{a}{b} \sim 1 + 0.318\theta_0^{4/3} - 0.15\theta_0^2 + \cdots,$$

whence

$$\frac{b}{a} \sim 1 - 0.518\theta_0^{4/3} + 0.15\theta_0^2 + \cdots,$$

Consequently, for $M = 1$, $\varkappa = 1.4$,

$$\left.\frac{b}{a}\right|_{5^\circ} = 0.9880, \qquad \left.\frac{b}{a}\right|_{10^\circ} = 0.9737, \qquad \left.\frac{b}{a}\right|_{15^\circ} = 0.9547.$$

If $\varkappa = 2$, then

$$\left.\frac{b}{a}\right|_{5^\circ} = 0.9882, \qquad \left.\frac{b}{a}\right|_{10^\circ} = 0.9739, \qquad \left.\frac{b}{a}\right|_{15^\circ} = 0.9555.$$

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Note: Figure translations are in progress. See original paper for figures.