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Abstract

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MATHEMATICS

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ASYMPTOTICS OF SOLUTIONS OF CERTAIN BOUNDARY-VALUE PROBLEMS WITH OSCILLATING BOUNDARY CONDITIONS

In ⁽¹⁾ an asymptotic formula was given for the solutions of the first boundary-value problem for elliptic equations with rapidly oscillating boundary conditions (there a definition of $\frac{1}{\varepsilon}$ -oscillating functions is also given); it was shown that in this case the “second iteration process” of the work ⁽²⁾ takes place and that a boundary effect occurs. In the present work we shall solve analogous problems for certain other types of equations. We restrict ourselves to the case of only two spatial variables and locally prescribed boundary conditions having the form of a simple harmonic multiplied by a smooth function that vanishes outside a neighborhood of a certain point. The transition to more general boundary conditions and to the n -dimensional case is carried out in the same way as in ⁽¹⁾. At the same time, below we describe only the first step of the “second iteration process”; the subsequent steps and estimates of the remainder terms are carried out as in ^(1,2). We note that the boundary effects described here occur, for example, in the phenomenon of the skin effect, in the propagation of surface waves, etc.

I. Parabolic equations. Consider in the space (x, y, t) a parabolic equation of the form

$$\frac{\partial u}{\partial t} + L_{2k}u = 0, \quad (1)$$

where L_{2k} is an elliptic operator in the variables (x, y) of order $2k$ with sufficiently smooth coefficients, which may depend on x, y and t . Let Q be a domain in the (x, y) -plane with smooth boundary Γ ; Ω is the cylinder $Q \times T$ ($T = (-\infty < t < +\infty)$) with boundary $\Gamma_1 = \Gamma \times T$. Introduce in a neighborhood of Γ coordinates (ρ, φ) ⁽²⁾, where φ is the coordinate on Γ ; ρ is the distance from Γ along the normal. Consider the solution of equation (1) under the boundary conditions

$$\left. \frac{\partial^s u}{\partial n^s} \right|_{\Gamma_1} = A_s(\varphi, t)e^{i(\omega t + \gamma \varphi)} \quad (s = 0, 1, \dots, k-1), \quad (2)$$

where $A_s(\varphi, t)$ is a function smooth on Γ_1 and vanishing outside a certain neighborhood. We shall seek the asymptotics of the solution of problem (1), (2) in the form

$$u = v(\rho, \varphi, t)e^{i(\omega t + \gamma\varphi)}.$$

Passing in equation (1) to the variables (ρ, φ) and expanding the coefficients of the resulting equation in powers of ρ ^(1,2), then, as in formula (5) of ⁽¹⁾, we obtain $L_{2k}u = \exp i(\omega t + \gamma\varphi) [M_{2k}(\frac{\partial}{\partial\rho}, i\gamma) + \dots] v$ and equation (1), after division by $\exp i(\omega t + \gamma\varphi)$, takes the form

$$\left[i\omega + M_{2k} \left(\frac{\partial}{\partial\rho}, i\gamma \right) + \dots \right] v = 0; \quad (3)$$

$M_{2k}(\xi, \eta) = \sum a_p(\varphi, t)\xi^p\eta^{2k-p}$ is a homogeneous form of degree $2k$ with coefficients depending on φ and t , such that $M_{2k}(i\xi, i\eta) \geq c(\xi^{2k} + \eta^{2k})$, $c > 0$ ⁽¹⁾.

Let us consider three cases:

1. ω is of higher order than γ^{2k} , for example, $\gamma = O(\omega^{1/s})$, $s > 2k$ (in particular, $\gamma = 0$). Then, making the substitution $\tau = \omega^{1/2k}\rho$, we see that the principal part of the operator on the left-hand side of (3) will be $i\omega + a_0\partial^{2k}/\partial\rho^{2k} = \omega[i + a_0\partial^{2k}/\partial\tau^{2k}]$, $(-1)^k a_0 > 0$, and the first approximation to equation (3) will be the equation

$$a_0 \frac{\partial^{2k} v}{\partial \tau^{2k}} + iv = 0. \quad (4)$$

Its characteristic equation $a_0\lambda^{2k} + i\omega = 0$ has exactly k roots: $-\lambda_1, \dots, -\lambda_k$ with negative real parts. The general solution v_0 of equation (4) of boundary-layer type will be

$$v_0 = \sum_{i=1}^k c_i(\varphi, t) \exp(-\lambda_i \tau) = \sum_{i=1}^k c_i(\varphi, t) \exp(-\lambda_i \omega^{1/2k} \rho). \quad (5)$$

2. $\gamma = \alpha\omega^{1/2k}$, $\alpha = O(1)$. Making the substitution $\tau = \omega^{1/2k}\rho$, in the first approximation we replace equation (3) by the equation

$$[i + M_{2k}(\partial/\partial\tau, i\alpha)]v = 0. \quad (6)$$

This is an ordinary differential equation with coefficients constant with respect to τ , and its characteristic equation $i + M_{2k}(\lambda, i\alpha) = 0$ has exactly k roots: $-\lambda_1, \dots, -\lambda_k$ with negative real parts ⁽²⁾. The general solution v_0 of equation (6) of boundary-layer type also has the form (5).

3. $\gamma = O(\omega^{1/s})$, $s < 2k$. In this case the substitution $\tau = \gamma\rho$ leads to the equation $M_{2k}(\partial/\partial\tau, i)v_0 = 0$, and again (cf. (1)) we obtain the solution v_0 of boundary-layer type of the form (5) (with $\omega^{1/2k}$ replaced by γ).

The coefficients c_i in (5) are determined uniquely from the boundary conditions (2). Having constructed the first approximation v_0 , we, as in (1, 2), construct the subsequent approximations by solving analogous elementary differential equations and estimating the remainder terms.

Thus, strong oscillation of the boundary conditions in t or in φ leads to a boundary effect. The asymptotics of a solution of any order can be obtained with the aid of the "second iteration process."

II. Hyperbolic equations. Consider, for example, the second-order hyperbolic equation

$$\frac{\partial^2 u}{\partial t^2} - L_2 u = 0 \quad (7)$$

in the cylinder Ω , with the boundary condition on the boundary Γ_1

$$u|_{\Gamma_1} = A(\varphi, t)e^{i(\omega t + \gamma\varphi)}. \quad (8)$$

Passing to the coordinates ρ, φ and expanding the coefficients in powers of ρ , we represent L_2 in the form $L_2 = M_2\left(\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\varphi}\right) + \dots$, where

$$M_2\left(\frac{\partial}{\partial\rho}, \frac{\partial}{\partial\varphi}\right) = a_0(\varphi, t)\frac{\partial^2}{\partial\rho^2} + 2a_1(\varphi, t)\frac{\partial^2}{\partial\rho\partial\varphi} + a_2(\varphi, t)\frac{\partial^2}{\partial\varphi^2}.$$

We shall seek the asymptotics of the solution of problem (7), (8) in the form $u = v(\rho, \varphi, t) \times \exp[i(\omega t + \gamma\varphi)]$. We obtain for the first approximation v_0 to v the equation

$$\left[-\omega^2 - M_2\left(\frac{\partial}{\partial\rho}, i\gamma\right)\right]v_0 = (-\omega^2 + a_2\gamma^2) - 2i\gamma a_1\frac{\partial v}{\partial\rho} - a_0\frac{\partial^2 v}{\partial\rho^2} = 0. \quad (9)$$

If $D\gamma^2 - a_0\omega^2 = q > 0$, where $D = a_0a_2 - a_1^2$, then among the roots of the characteristic equation (6) there will be one, $-\lambda_1\gamma$, with negative real part. It corresponds to a boundary-layer-type solution of equation (9) of the form $v_0 = C(\varphi, t)\exp(-\lambda_1\gamma\rho)$, where C_1 is determined from the boundary condition (8): $C_1 = A$. Thus, for $q > 0$ a boundary effect is observed.

The constructions given are also generalized to equations

$$\frac{\partial^2 u}{\partial t^2} + L_{2k}u = 0,$$

where L_{2k} is an elliptic operator, under boundary conditions (2), if the corresponding boundary-value problem for them is correct (for example, L_{2k} is a self-adjoint operator).

III. Degeneration of elliptic operators into elliptic ones. Such a problem was studied by L. A. Gol' denveizer ⁽³⁾ in application to the theory of thin elastic shells.

Consider, as in ⁽²⁾, §§ 6-7, the elliptic differential equation

$$L_\varepsilon u \equiv \sum_{s=0}^{2l} \varepsilon^s L_{2k+s} u = 0, \quad (10)$$

for which the first boundary-value problem is always solvable (see ⁽²⁾, § 7). Let for the corresponding generalized characteristic form π_ε we have $\operatorname{Re} \pi_\varepsilon(i\xi, i\eta) > 0$.

We prescribe the boundary conditions in the form

$$\left. \frac{\partial^s u}{\partial n^s} \right|_\Gamma = A_s(\varphi) e^{i\omega\varphi}. \quad (11)$$

Here it is necessary to distinguish three cases of the relationship between ω and ε , as Gol' denveizer did in ⁽³⁾, and in all cases, for any ω , the phenomenon of a boundary layer occurs.

We shall seek the solution of (10), (11) in the form $u = v \exp i\omega\varphi$. For this purpose, near the boundary we pass to the coordinates (ρ, φ) and expand the coefficients in powers of ρ . We obtain

$$L_\varepsilon u = \sum_{s=0}^{2l} \varepsilon^s H_{2k+s} \left(\frac{\partial}{\partial \rho}, i\omega \right) + \dots = 0,$$

where $H_{2k+s}(\xi, \eta)$ is a homogeneous polynomial of order $2k + s$.

In the case when $1/\varepsilon$ is of higher order than ω , the construction of the boundary layer is the same as in ⁽²⁾, § 7.

In the case when they have the same order: $\omega\varepsilon = O(1) = \alpha$, putting $\omega\rho = \tau$, we obtain the first approximation v_0 to v by solving an ordinary equation with coefficients constant with respect to τ :

$$\sum_{s=0}^{2l} \alpha^s H_{2k+s} \left(\frac{\partial}{\partial \tau}, i \right) v_0 = 0.$$

The corresponding characteristic equation, by virtue of Lemma 4 of ⁽²⁾, has exactly k roots: $-\lambda_1, \dots, -\lambda_k$, with negative real parts, and the general solution v_0 of boundary-layer type will be

$$v_0 = \sum_i^k c_i(\varphi) \exp(-\lambda_i \omega \rho).$$

The coefficients c_i are determined from the boundary conditions (11).

In the case where $1/\varepsilon$ is of lower order than ω , the first approximation v_0 is determined from the equation $H_{2(k+l)}(\partial/\partial\rho, i\omega) = 0$, and the solution v_0 of boundary-layer type has the previous form.

IV. Degeneration of a hyperbolic equation into a parabolic one. For simplicity, we shall seek in Ω a solution of an equation of the form

$$\varepsilon \frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - L_2 u = 0$$

under the boundary condition (8), where ε is small together with $1/\omega$; L_2 is a second-order elliptic operator.

Introducing, as above, the variables ρ, φ and setting $u = v(\rho, \varphi) \exp i(\omega t + \gamma\varphi)$, we obtain for v , in the first approximation,

$$\left[-\varepsilon\omega^2 + i\omega - M_2 \left(\frac{\partial}{\partial\rho}, i\gamma \right) \right] v = 0. \quad (12)$$

Here too one must distinguish three cases:

In the first case $\omega \gg \gamma^2$. Then, as the first approximation, one may take the solution of the simpler equation

$$\left(-\varepsilon\omega^2 + i\omega - a_0 \frac{\partial^2}{\partial\rho^2} \right) v = 0. \quad (13)$$

The corresponding characteristic equation, for small ε and large ω , has one root $-\lambda_1$ with a large negative real part, and $v_0 = c_1(\varphi) \exp(-\lambda_1\rho)$; v_0 is a solution of boundary-layer type.

In the other cases, i.e. when $\omega = O(\gamma^2)$ or when $\omega = o(\gamma^2)$, the form of equation (13), which determines the first approximation v_0 , changes, but, as before, it admits a solution of boundary-layer type. The transition to the subsequent approximations is carried out as usual.

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Note: Figure translations are in progress. See original paper for figures.

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