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# MATHEMATICS

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**Abstract**

**Full Text**

## MATHEMATICS

**A. I. Kostrikin**

### ON THE BURNSIDE PROBLEM

*(Presented by Academician I. M. Vinogradov on 4 III 1958)*

Does there exist, among finite groups with a given number  $q$  of generators and the identity relation  $x^p = 1$  ( $p$  prime), a group of some maximal order depending only on  $q$  and  $p$ ?

This question, known as the weakened Burnside problem for prime exponent  $p$ , has long attracted attention both because of the prospects it opens in the classification of finite  $p$ -groups and because it is part of the general group-theoretic problem of the coincidence of classes of locally finite groups, on the one hand, and periodic groups, on the other <sup>(1)</sup>. An answer to it has been obtained, however, only for certain special values of  $p$ . The author solved the weakened Burnside problem first for  $q = 2$  and  $p = 5$  <sup>(2)</sup> (the first nontrivial case), and then for arbitrary  $q$  for  $p = 5$  <sup>(3)</sup> and  $p = 7$  <sup>(4)</sup>. Groups with  $p = 5$  were also considered (independently and at approximately the same time as <sup>(3)</sup>) by Higman <sup>(5)</sup>.

In the present note the weakened Burnside hypothesis is proved for all prime exponents  $p$ .

Here, as in the works cited above, use is made of the connection, established earlier by a number of authors (for the literature see <sup>(2,4)</sup>), between periodic groups and Lie rings, namely the fact that the weakened Burnside problem for prime exponent  $p$  is a special case (since what is at issue is its positive solution) of the following problem of "Burnside type" from the theory of Lie rings. Will a Lie ring of characteristic  $p$  be nilpotent if it has a finite number of generators and satisfies the  $n$ -th Engel condition,  $n < p$ , i.e., if for any two of its elements the relation

$$[\dots [[uv]v] \dots v] = [uv^n] = 0$$

is satisfied?

In our note <sup>(6)</sup> it was reported that local nilpotency had been proved for Lie rings with the  $n$ -th Engel condition and characteristic  $p > n + [n/2]$  (or  $p = 0$ ). It was noted there that extending this result to a broader class of Lie rings with characteristic  $p > n$  is connected with finding, in every such ring  $\mathfrak{R}$ , an element  $c_{(2)}$ —the second term of a sequence  $[c_{(1)}, c_{(2)}, \dots, c_{(m)}, \dots]$ , characterized by the

relations:

$$c_{(m)} \neq 0, \quad c_{(m)} u^\alpha c_{(m)} = 0, \quad \alpha = 0, 1, \dots, 2m-1, \quad 1 \leq m \leq \left\lfloor \frac{n-1}{2} \right\rfloor, \quad u \in \mathfrak{A}.$$

A priori it is not clear whether even the element  $c_{(1)}$  actually exists. It is known already from <sup>(4)</sup> that, together with  $c_{(m)}$ , any monomial of the form  $c'_{(m)} = [c_{(m)} u^{2m+1} c_{(m)}]$  satisfies the same identities. If, however,  $c'_{(m)} = 0$  for every  $u$  in  $\mathfrak{A}$ , then  $c_{(m)}$  may be taken as the element  $c_{(m+1)}$ . This circumstance suggests constructing the sequence  $c_{(1)}, c_{(2)}, \dots$  recursively, by “iterating” each separate term with the aid of the preceding one.

It is proved that for  $m > 2$  the indicated path very quickly leads to the goal: the element  $c_{(m+1)}$  is found in the triple

$$c_{(m)}, \quad c'_{(m)} = [c_{(m)} a^{2m+1} c_{(m)}], \quad [c'_{(m)} b^{2m+1} c'_{(m)}].$$

Since the principal ideal  $\mathfrak{A} = \{c_{\lfloor \frac{n-1}{2} \rfloor}\}$  is commutative, the precise meaning of this assertion is as follows. If in an Engel Lie ring  $\mathfrak{L}$  of characteristic  $p > n$  there is an element  $c_{(2)}$ , then  $N(\mathfrak{L}) \neq 0$ . Here  $N(\mathfrak{L})$  is the radical of the ring  $\mathfrak{L}$ , or the sum of its locally nilpotent ideals.

Without the additional restrictions imposed in [6] on the characteristic of the ring, the proof of the existence of the element  $c_{(2)}$  requires considerable effort, and in our brief exposition it is not always easy to trace. However, one may note that over all the arguments there dominates the method of iteration, understood only in a broader sense than before. The elements constructed are almost every time replaced by others; at the same time the properties of the former are not only reproduced, but are also gradually improved in the desired direction. Let us add also that the proof of the main theorem proceeds by contradiction; therefore, for convenience in Theorems 1 and 2, by  $\mathfrak{L}$  one should understand a nonzero Engel ( $n < p$ ) ring without radical, isomorphically represented by its ring of inner derivations. This convention justifies a certain liberty in our handling of Lie elements in the corresponding enveloping ring  $\mathfrak{A}_{\mathfrak{L}}$ .

**Theorem 1.** *In any Engel Lie ring  $\mathfrak{L}$  of characteristic  $p > n$  one can find a nonzero element  $c$  of second order:  $c^2 = 0$ .*

Indeed,  $u^n = 0$  for all  $u$  from  $\mathfrak{L}$ .

Let  $v^m = 0$ ,  $4 \leq m \leq n$ . Without special difficulty one verifies the identity  $[uv^{m-1}]^{m-1} = 0$ . After a finite number of such descents we arrive at an element  $b \neq 0$  of third order:  $b^3 = 0$ . It is easy to see that  $[ub^2]^2 = b^2 u^2 b^2$ , and therefore our goal may be considered attained if  $b^2 u^2 b^2 = 0$ . The role of the desired element  $c$  will then be played either by  $b$ , or by some monomial  $[ab^2] \neq 0$ . But as yet we do not even know how long the nonzero products of the form  $b^2 (u^2 b^2)^k$  may be. Introduce the notation:  $g_m(u) = [u[bu]^m b^2]$ , where  $u$  is an arbitrary element of  $\mathfrak{L}$ . It turns out that the relation holds

$$(g_m(u))^2 = b^2(u^2b^2)^{m+1}.$$

Since  $g_{n-1}(u) = (-1)^{n-1}[b[bu]^n] = 0$ , this gives the answer to the question posed above:  $b^2(u^2b^2)^t = 0$ ,  $t \leq n$ .

We shall now seek such elements of third order for which the last identity would hold with ever lower exponent  $t$ . Let  $g_m(f) \neq 0$ , but  $g_{m+1}(f) = 0$  ( $f$  is some fixed element). Putting  $a = [f[bf]^m]$  and  $b_0 = [ab^2] \neq 0$ , we easily discover a relation between the elements  $b_0$ ,  $a$ , and  $b$  which gives the key to the induction:  $[b_0ab] = 0$ . Simple arguments based on the use of this relation show that  $b_0^2(u^2b_0^2)^s = 0$ ,  $s = [t/2]$ . Repeating such a procedure several times, we arrive, in the end, at an element  $b_\nu$  of third order which satisfies the identity  $b_\nu^2u^2b_\nu^2 = 0$ .

Theorem 1 is proved.

**Theorem 2.** *In any Engel ring  $\mathfrak{L}$  of characteristic  $p > n$  there exists an element  $c_{(2)}$ .*

**Proof.** We first make a small observation. If  $cu^\alpha c = 0$ ,  $\alpha = 0, 1, \dots, 2k$ , then also  $cu^{2k+1}c = 0$ . In particular, the property of  $c$  being an element of second order also means that  $cuc = 0$ , and the element  $c_{(2)}$  is completely determined by the identities:  $c_{(2)}^2 = c_{(2)}u^2c_{(2)} = 0$ .

We shall start from the element  $c_{(1)} = c$ , whose existence is ensured by Theorem 1. If  $cu^2c = 0$ , then, in accordance with the remark made, the proof ends here.

- 1) One may adopt a somewhat more general point of view and take interest in the consequences of the relation

$$A_m = cu_1^2cu_2^2c \dots cu_m^2c = 0$$

for some finite index  $m$  and arbitrary  $u_i \in \Omega$ .

In this case the following fact holds. If  $A_{m-1} \neq 0$ , but  $A_m = 0$ , then

$$c_0^2 = c_0u^2c_0v^2c_0 = 0, \quad \text{where } c_0 = [ca_1^3ca_2^2c \dots ca_{m-1}^2c] \neq 0.$$

And further: the last nonzero term in the sequence  $c_0$ ,  $c_{i+1} = [c_i a_i^3 c_i]$ ,  $i = 0, 1, 2$ ;  $a_i \in \Omega$ , may be taken as the element  $c_{(2)}$ .

It remains to assume that there exists a product  $A_m \neq 0$  with an arbitrarily large number  $m$ .

- 2) This circumstance, as can be proved, leads us to the following conclusion. There are two elements  $c_i = [ca_i^3c]$ ,  $a_i \in \Omega$ ,  $i = 1, 2$ , which satisfy the conditions

$$c_1^2 = c_2^2 = [c_1c_2] = 0, \quad c_1c_2 \neq 0.$$

- 3) Beginning from this point, we forget the connection of  $c_i$  with the initial element  $c$ . We shall consider, in general, the set  $M$  of all pairs of elements

$c_1, c_2 : c_1 c_2 \neq 0, c_1^2 = c_2^2 = [c_1 c_2] = 0$ . Suppose that the subset  $S \subset M$ , singled out by the additional identities  $c_1 u^2 c_1 c_2 = c_2 u^2 c_2 c_1 = 0$ , is empty. Then in  $\Omega$  there exists an element  $c_{(2)}$ . Clarifying the details of the proof of this assertion would take us far aside.

4) Thus, in  $\Omega$  there exist elements  $c_1, c_2$  subject to the following requirements:

$$c_1 c_2 \neq 0, \quad c_1^2 = c_2^2 = [c_1 c_2] = c_1 u^2 c_1 c_2 = c_2 u^2 c_2 c_1 = 0, \quad u \in \Omega.$$

It can be shown that  $c_1^{(1)} c_2^{(1)} \neq 0$ ,

$$(c_1^{(1)})^2 = (c_2^{(1)})^2 = [c_1^{(1)} c_2^{(1)}] = c_1^{(1)} u^\alpha c_1^{(1)} c_2^{(1)} = 0$$

( $\alpha = 0, 1, 2, 3$ ), where  $c_1^{(1)} = [f_1 c_1 c_2]$ ,  $c_2^{(1)} = [c_1 h_1^3 c_1]$ , and  $f_1, h_1$  are certain fixed elements of  $\Omega$ . The most difficult point here is the justification of the inequality  $c_1^{(1)} c_2^{(1)} \neq 0$ .

Putting again  $c_1^{(2)} = [f_2 c_1^{(1)} c_2^{(1)}]$ ,  $c_2^{(2)} = [c_1^{(1)} h_2^3 c_1^{(1)}]$ , we arrive at elements which satisfy the identities

$$(c_1^{(2)})^2 = (c_2^{(2)})^2 = [c_1^{(2)} c_2^{(2)}] = c_1^{(2)} u^\alpha c_1^{(2)} c_2^{(2)} = c_2^{(2)} u^\alpha c_1^{(2)} c_2^{(2)} = 0 \quad (\alpha = 2, 3)$$

and the condition  $c_1^{(2)} c_2^{(2)} \neq 0$ .

After a finite number of transformations of the same type we obtain elements (denote them again by  $c_1, c_2$ ) whose properties

$$c_1 c_2 \neq 0, \quad c_1^2 = c_2^2 = [c_1 c_2] = c_1 u^\alpha c_1 c_2 = c_2 u^\alpha c_2 c_1 = 0, \quad \alpha \geq 0,$$

suggest the way to apply them.

5) Consider in  $\Omega$  the principal ideal  $\mathfrak{N} = \{c_0\}$ ,  $c_0 = [ae^2] \neq 0$ ,  $e = c_1 + c_2$ , where  $c_1$  and  $c_2$  are taken from item 4).

By direct verification we ascertain that

$$c_0 h^2 c_0 g^2 c_0 = 0$$

for all  $h, g \in \mathfrak{N}$ .

From item 1) follows the existence in  $\mathfrak{N}$  of an element  $\tilde{c}_{(2)}$ , and also (as is evident from the remarks made above concerning the results of the work (6)) of an element  $\tilde{c}_{(3)}$ , if  $p > 7$ . In any case we obtain an element  $\tilde{c} \in \mathfrak{N}$  satisfying the relations

$$\tilde{c} h^\alpha \tilde{c} = 0, \quad 0 \leq \alpha \leq 5, \quad h \in \mathfrak{N},$$

i.e.

$$[g \tilde{c} h^\alpha \tilde{c}] = 0$$

for any  $g \in \mathfrak{N}$ .

As is easy to see,

$$c_0^2 = [ae^2]^2 = 0$$

in  $\Omega$ , and  $\tilde{c}$  is obtained from  $c_0$  as a result of the application of such simple operations that the property of being an element of second order is not lost by them. It is comparatively simple to show that,

that in  $\mathfrak{L}$  the relations

$$(\tilde{c}_0)^2 = \tilde{c}_0 h_1 h_2 \tilde{c}_0 = 0, \quad h_1, h_2, \tilde{c}_0 \in \mathfrak{M},$$

hold, where  $\tilde{c}_0$  is either an initial element  $\tilde{c}$ , or a monomial  $[w\tilde{c}h^2\tilde{c}] \neq 0$ ,  $w \in \mathfrak{L}$ .

Taking, in particular,  $h_1 = [\tilde{c}_0 u^3]$ ,  $h_2 = [\tilde{c}_0 v^3]$ , we arrive at the identity

$$[\tilde{c}_0 u^3 \tilde{c}_0][\tilde{c}_0 v^3 \tilde{c}_0] = 0,$$

which, according to points 1) and 2), may be regarded as the completion of the proof of Theorem 2.

We now formulate the proposition that is the main goal of our investigation.

**Main Theorem.** *An arbitrary Lie ring  $\mathfrak{L}$  satisfying the  $n$ -th Engel condition and having characteristic  $p \geq n$  (or  $p = 0$ ) is locally nilpotent.*

The proof of the main part of the assertion, when  $p > n$ , follows from the preceding remarks and the two theorems, while the case  $p = n$  may be added on the basis of the simplest considerations.

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*Note: Figure translations are in progress. See original paper for figures.*

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