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Abstract

Full Text

MATHEMATICS

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DISPERSION OF DIVISORS AND QUADRATIC FORMS IN PROGRESSIONS AND SOME BINARY ADDITIVE PROBLEMS

Let
$\tau(n) = \tau_2(n) = \sum_{x_1 x_2 = n} 1$ be the number of divisors of n;
$\tau_k(n) = \sum_{x_1 x_2 \ldots x_k = n} 1$;
$Q(u,v)$ be an integral primitive positive binary quadratic form of discriminant $d < 0$. Sufficient information on the behavior of $\tau(n)$ and $Q(u,v)$ in arithmetic progressions makes it possible to solve certain interesting binary additive problems. The sums

$$\sum_{\substack{m \equiv l \pmod D \\ m \leq n}} \tau(m); \tag{1}$$

$$\sum_{\substack{Q(u,v) \equiv n \pmod D \\ Q(u,v) \leq n}} 1 \tag{2}$$

for large n and $D \leq n^{1/3-\varepsilon_0}$ can be treated by means of André Weil' s estimates for Kloosterman sums (in what follows $\varepsilon_i, \eta_i, \varepsilon_i$ are small positive constants). (Concerning (1), see ([1]), pp. 22-29, and ([2]).)

For such values of D the following formulas are obtained: for $(l, D) = 1,\ 1 \leq l < D$,

$$\sum_{\substack{m \equiv l \pmod D \\ m \leq n}} \tau(m) = n \ln n \cdot \frac{\varphi(D)}{D^2} \rho(n, D) + O\left(n^{1-\alpha_1}\right); \tag{3}$$

$$\sum_{\substack{Q(u,v) \equiv n \pmod D \\ Q(u,v) \leq n}} 1 = \frac{2\pi n}{\sqrt{|d|}\, D} \sigma(n, D) + O\left(n^{1-\alpha_1}\right), \tag{4}$$

where

$$\rho(n,D) = 1 - \frac{1}{\ln n}\left(1 - 2C_0 + 2\sum_{p|D}\frac{\ln p}{p-1}\right), \tag{5}$$

while $\sigma(n,D)$ is defined more complicatedly. Let $(n,D) = d_1;\ D = D_1 d_1;\ d_1 = d_{11}d_{12}$, where d_{11} consists of prime factors $p \mid D_1$, and $(d_{12}, D_1) = 1$. Further, let $X_d(m) = (d/m)$. Then

$$\sigma(n,D) = \prod_{p|D}\left(1 - \frac{X_d(p)}{p}\right) \times \prod_{p|d_{11}}\left(1 + X_d(p) + \dots + X_d^{a_p}(p)\right) \times$$

$$\times \prod_{p|d_{12}}\left(1 + X_d(p) + \dots + X_d^{a_p-1} + \frac{X_d^{a_p}(p)}{1 + X_d(p)/p}\right), \tag{6}$$

where $p^{a_p} \mid d_{1i};\ p^{a_p+1} \nmid d_{1i}$.

For some binary additive problems, however, formulas of this type are needed for $D \le n^{1-\eta_0}$. Then estimates of André Weil type no longer achieve the goal, and the corresponding formulas are unknown. However, for

for solving the corresponding binary problems it suffices to have formulas of the type (3) and (4) not for all large D, but for "almost all" of them, which leads to the idea of considering the "dispersion" of the number of divisors and of the values $Q(u,v)$ in arithmetic progressions. These considerations lead to the formulas:

Theorem 1. Let $n^{1/2} \ll D_1 \ll n^{1-\eta_1}$, $D_2 = D_1^{1-\eta_2}$, $l \leqslant n$. Then

$$\sum_{\substack{D_1 \leqslant D \leqslant D_1 + D_2 \\ (D,l)=1}} \left(\sum_{\substack{m \equiv l \pmod D \\ m \leqslant n}} \tau(m) - \frac{n \ln n \cdot \varphi(D)}{D^2}\rho(n,D)\right)^2 = O\left(\frac{n^{2-\eta_3}D_2}{D_1^2}\right). \tag{7}$$

Theorem 2.

$$\sum_{D_1 \leqslant D \leqslant D_1 + D_2} \left(\sum_{\substack{Q(u,v) \equiv n \pmod D \\ Q(u,v) \leqslant n}} 1 - \frac{2\pi n}{\sqrt{|d|}\,D}\sigma(n,D)\right)^2 = O\left(\frac{n^{2-\eta_3}}{D_1^2}D_2\right), \tag{8}$$

where D_1 and D_2 are defined as before.

These theorems give the "dispersion" of the quantities of interest to us in arithmetic progressions. Applying the obvious analogue of Chebyshev' s inequality, known in probability theory, we can conclude that "almost for all" D between D_1 and $D_1 + D_2$ the formulas (3) and (4) will hold. The number of exceptional D for which this will not be satisfied is equal to $O(D_1^{1-\eta_4})$, and for $D \ll \sqrt{n}$ we also apply (3) and (4). In solving the corresponding binary additive problems, the influence of the exceptional D is taken into account by means of a trivial upper estimate.

One of the problems that admits such a treatment is the problem of an asymptotic expression for the sum

$$\sum_{m \leqslant n} \tau_2(m)\tau_k(m+l), \tag{9}$$

which has been studied by many English mathematicians. For $k = 2$ such an expression is given in ([6]); for $k = 3$, in a recent work of C. Hooley ([2]). For $k > 3$ the methods set out there are inapplicable.

The sum (9) coincides with the number of solutions of the Diophantine equation

$$x_1x_2 - y_1y_2 \cdots y_k = l, \qquad 1 \leqslant x_1x_2 \leqslant n. \tag{10}$$

With the aid of Theorem 1 one succeeds in proving a theorem valid for any k; for simplicity of formulation we take $l = 1$.

Theorem 3. As $n \to \infty$, $k \geqslant 2$,

$$\sum_{m \leqslant n} \tau_2(m)\tau_k(m+1) \sim k!\, C_{k-1} S_k n (\ln n)^k, \tag{11}$$

where

$$S_k = \sum_{n=1}^{\infty} \mu(n) n^{-2} \times \prod_{p|n} \left(p \left(1 - \left(1 - \frac{1}{p} \right)^{k-1} \right) \right);$$

$$C_{k-1} = \lim_{Y \to \infty} (\ln Y)^{-k+1} \int \cdots \int_{\substack{1 \leqslant y_1 \leqslant \cdots \leqslant y_{k-1} \leqslant Y \\ (y_1 \cdots y_{k-1})^{-1}}} (y_1 \cdots y_{k-1})^{-1} \, dy_1 \cdots dy_{k-1}.$$

The same method makes it possible to obtain for the left-hand side of (11) an asymptotic expansion of the form

$$nP_k(\ln n) + O(n^{1-\zeta_0}), \tag{12}$$

where P_k is a polynomial of degree k.

Let $Q(u,v)$ be the binary form of discriminant $d<0$ mentioned earlier; let $\mathfrak{K}$ be a field of algebraic numbers, Abelian over the field of rational numbers, of discriminant $d_{\mathfrak{K}}$; let $g>0$ be an integer, and suppose that $(d, gd_{\mathfrak{K}})=1$.

Theorem 4. For $n>n_0(d,g,d_{\mathfrak{K}})$, the equation

$$n = Q(u,v) + gN(\mathfrak{a}), \tag{13}$$

where $N(\mathfrak{a})$ is the norm of an integral ideal from $\mathfrak{K}$, is solvable. The corresponding asymptotic law holds.

In the special case $\mathfrak{K}=k(\sqrt{-1})$, $N(\mathfrak{a})=z^2+t^2$, we obtain the already known theorem in the theory of quaternary quadratic forms: $n=Q(u,v)+g(z^2+t^2)$. Another special case is obtained by a different special choice of $\mathfrak{K}$.

Theorem 5. For a given $D>0$ and $n>n_0(D)$, the equation

$$n = \Pi_1 + \Pi_2, \tag{14}$$

is solvable, where Π_1 is a product of primes $\equiv 1 \pmod 4$, and Π_2 is a product of primes $\equiv 1 \pmod D$. If p/Π_1 is counted twice, and p/Π_2 $\varphi(D)$ times, then one can give an asymptotic formula for the number of solutions.

For $D=4$ one obtains the known theorem on four squares. For the special case when the Abelian field has a zeta-function of the form $A(s)=\zeta(s)\prod_{k=1}^{q-1}L(s,\chi_k)$, where the χ_k correspond to power residues modulo a prime modulus, and when $g=1$, the asymptotic formula for the number of solutions of (13) has the form

$$\frac{2\pi n}{\sqrt{|d|}}\,\mathfrak{S}(n) + O(n^{1-\zeta_1}), \tag{15}$$

where

$$\mathfrak{S}(n) = \sum_{\delta_1,\delta_2,\dots,\delta_{q-1}=1}^{\infty} \frac{\chi_1(\delta_1)\cdots\chi_{q-1}(\delta_{q-1})}{\delta_1\delta_2\cdots\delta_{q-1}}\,\sigma(n,\delta_1\delta_2\cdots\delta_{q-1}) \tag{16}$$

($\sigma(n,D)$ is taken from (6)). The series $\mathfrak{S}(n)$ is conditionally convergent, and $\mathfrak{S}(n)>1/\ln n$ for large n.

The derivation of Theorems 1 and 2 is based on the theory of representation of large numbers by ternary quadratic forms, developed by the author and A. V. Malyshev (see, for example, [1,3–5]). Here it is possible only to explain it briefly. If one expands the brackets in the left-hand side of (7), sums of terms of the form $\tau(D\nu_1+l)\tau(D\nu_2+l)$ are obtained. If $(D\nu_1+l, D\nu_2+l)=1$, then this product is equal to $\tau(D^2\nu_1\nu_2+Dl(\nu_1+\nu_2)+l^2)$, and the latter is equal to the number of solutions of the equation $D^2\nu_1\nu_2+Dl(\nu_1+\nu_2)+l^2=vw$, which, after summing over D, can be reduced to the representation of the number $(l(\nu_1-\nu_2))^2$ by the form u^2-vw under suitable geometric and congruence conditions (8); the case $(D\nu_1+l, D\nu_2+l)>1$ is treated analogously.

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REFERENCES

[1] Yu. V. Linnik, *Vestn. LGU*, No. 5 (1955).
[2] C. Hooley, *Proc. Lond. Math. Soc.*, 7, No. 27 (1957).
[3] Yu. V. Linnik, *Izv. AN SSSR, ser. matem.*, No. 1 (1939).
[4] Yu. V. Linnik, *Usp. matem. nauk*, 4, (33) 5 (1949).
[5] Yu. V. Linnik and A. V. Malyshev, *Usp. matem. nauk*, 8, No. 5 (1953).
[6] A. E. Ingham, *J. Lond. Mat. Soc.*, No. 2 (1927).

Note: Figure translations are in progress. See original paper for figures.