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Abstract

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ON THE DISTRIBUTION OF NODES IN THE INTERPOLATION PROCESS OF S. N. BERNSTEIN

(Presented by Academician S. N. Bernstein on 26 XII 1957)

1°. Let a triangular matrix of nodes be given

$$-1 \leq x_1^{(n)} < x_2^{(n)} < \cdots < x_n^{(n)} \leq 1, \qquad n = 1, 2, \ldots, \tag{1}$$

and a function $f(x)$, defined on the segment $[-1, 1]$. According to a classical result of S. N. Bernstein and G. Faber, there exists no matrix of the form (1) for which the Lagrange interpolation process $L_n(f, x)$ satisfies, for every continuous function, uniformly, the relation

$$L_n(f, x) \to f(x), \qquad n \to \infty, \quad -1 \leq x \leq 1.$$

In connection with this negative result, the interpolation process $\{A_n(f, x)\}_{n=1}^{\infty}$ of S. N. Bernstein (^{1}), which is defined by the rule

$$A_n(f, x) = \sum_{j=1}^{n}{}' f(x_j^{(n)}) r_j(x), \qquad r_j(x) = l_j(x) + (-1)^{j-1} l_{2pt_j}(x), \tag{2}$$

is of interest. Here $\{l_j(x)\}_{j=1}^{n}$ are the fundamental Lagrange polynomials of the n-th row of the matrix (1); p is an arbitrary fixed natural number. The integer t_j is uniquely determined by the inequalities $2p(t_j - 1) < j < 2pt_j$. The prime on the sum (2) indicates that j takes all integer values from 1 to n, except the numbers divisible by $2p$. If $2pt_j > n$, then we set $l_{2pt_j}(x) = 0$.

The polynomials $A_n(f, x)$ have two important properties:

1) One can specify a very broad class of node matrices of the form (1) for which, for every function $f(x)$ continuous on $[-1, 1]$, the uniform relation ($^{1-3}$)

$$A_n(f,x) \to f(x), \qquad n \to \infty, \quad -1 \le x \le 1. \tag{3}$$

holds.

2) The ratio of the degree of the polynomial $A_n(f,x)$ to the number of its nodes is arbitrarily close to 1, provided only that p is sufficiently large.

On the other hand, in the case of equally spaced nodes of the interval $[-1,1]$, the process $A_n(f,x)$, constructed for the function $f(x) \equiv x$, diverges at all points $x \ne 0$ of the interval $(-1,1)$ ([1,4]).

In connection with the facts indicated, a natural question arises: what must the node matrix (1) be like in order that, for every continuous function, the relation (3) should hold uniformly? The present note is devoted to this question.

2°. Put

$$x_k^{(n)} = \cos\theta_k^{(n)}, \qquad k = 1,2,\dots,n, \quad n = 1,2,\dots.$$

We shall say that the nodes $\{x_k^{(n)}\}_{k=1}^n$, $n = 1,2,\dots$, are quasiuniformly distributed on the semicircle if there exist positive constants c_1 and c_2, independent of n, such that

$$\frac{c_1}{n} \le \theta_\nu^{(n)} - \theta_{\nu+1}^{(n)} \le \frac{c_2}{n}, \qquad \nu = 1,2,\dots,(n-1), \quad n = 2,3,\dots.$$

Theorem 1. *Suppose that the fundamental polynomials of S. N. Bernstein are bounded in the aggregate,*

$$\left|r_j^{(n)}(x)\right| \le c_3, \qquad j = 1,2,\dots, \quad n = 1,2,\dots, \quad -1 \le x \le 1.$$

Then the nodes of the polynomials $\{A_n(f,x)\}_{n=1}^\infty$ are quasiuniformly distributed on the semicircle.

From Theorem 1 it is easy to obtain Theorem 2.

Theorem 2. *In order that the interpolation process $\{A_n(f,x)\}_{n=1}^\infty$ of S. N. Bernstein, constructed for every function $f(x)$ continuous on the segment $[-1,1]$, converge uniformly on $[-1,1]$ to the function $f(x)$, it is necessary that the nodes of the polynomials $\{A_n(f,x)\}_{n=1}^\infty$ be quasiuniformly distributed on the semicircle.*

3°. In ([2]) sufficient conditions were given for relation (3) to hold uniformly for every continuous function. In particular, it was required of the node matrix that it satisfy the conditions

$$\left|l_1^{(n)}(1)\right| \le \left|l_2^{(n)}(1)\right| \le \dots \le \left|l_n^{(n)}(1)\right|, \qquad n = n_0, n_0+1, \dots; \tag{4}$$

$$\left|l_1^{(n)}(-1)\right| \geq \left|l_2^{(n)}(-1)\right| \geq \cdots \geq \left|l_n^{(n)}(-1)\right|, \qquad n = n_0, n_0+1, \ldots. \tag{5}$$

Inequalities (4) and (5) can be characterized very simply geometrically if we introduce into consideration the so-called conjugate points. Put

$$P_k(x) = l_k(x) + l_{k+1}(x), \qquad k = 1, 2, \ldots, (n-1).$$

$P_k(x)$ is a polynomial of degree $n-1$. The points $x_1, x_2, \ldots, x_{k-1}, x_{k+2}, \ldots, x_n$ are simple roots of $P_k(x)$. Since all roots of $P_k(x)$ are real, there exists a root X_k of the polynomial $P_k(x)$ distinct from its roots indicated above.

It is easy to see that

$$X_k = x_k + \frac{1}{1-a_k}(x_{k+1} - x_k), \qquad a_k = -\frac{\omega_n'(x_k)}{\omega_n'(x_{k+1})} > 0,$$

where

$$\omega_n(x) = \prod_{j=1}^{n}(x - x_j^{(n)}).$$

We shall call the point X_k conjugate with respect to the point x_k. If $a_k = 1$, then we set $X_k = \infty$. The conjugate points $\{X_k\}_{k=1}^{(n-1)}$, unlike the points $\{x_k\}_{k=1}^{n}$, generally speaking do not lie in the interval $(-1, 1)$. As will be seen from what follows, in interpolation theory the case when the conjugate points lie outside the interval $(-1, 1)$ is of special interest. Thus, for example, in the case of the nodes of P. L. Chebyshev

$$x_k = \cos\frac{2n - 2k + 1}{2n}\pi, \qquad k = 1, 2, \ldots, n,$$

the conjugate points are computed by the formulas

$$X_k = -\frac{\cos\frac{\pi}{2n}}{\cos\frac{k}{n}\pi}, \qquad k = 1, 2, \ldots, (n-1).$$

From these equalities it is seen that if $0 < x_k < 1$, then $X_k > 1$; while if $-1 < x_k < 0$, then $X_k < -1$. Thus, in the case of the P. L. Chebyshev nodes, the conjugate points lie outside the segment $[-1, 1]$.

Since the Chebyshev nodes satisfy inequalities (4) and (5), the question arises as to how the conjugate points are situated in the case of arbitrary node matrices satisfying inequalities (4) and (5). The answer is given by Theorem 3.

Theorem 3. *In order that the inequalities*

$$|l_k(1)| \leq |l_{k+1}(1)|, \qquad |l_k(-1)| \geq |l_{k+1}(-1)|,$$

hold, it is necessary and sufficient that the conjugate point X_k *satisfy the inequality* $X_k \leq -1$*, if* $a_k > 1$*, or the inequality* $X_k \geq 1$*, if* $a_k < 1$*.*

In (^{2}) it was proved that if the n-th row of the matrix (1) consists of the roots of the Jacobi polynomial $J_n(x, \alpha^{(n)}, \beta^{(n)})$ with parameters $-1 \leq \alpha^{(n)} \leq 0$, $-1 \leq \beta^{(n)} \leq 0$, then the matrix (1) satisfies inequalities (4) and (5). Therefore Theorem 4 follows from Theorem 3.

Theorem 4. *Let the* n*-th row of the matrix (1) consist of the roots of the Jacobi polynomial* $J_n(x, \alpha^{(n)}, \beta^{(n)})$*,* $n = 1, 2, \ldots$*, with parameters* $-1 \leq \alpha^{(n)} \leq 0$*,* $-1 \leq \beta^{(n)} \leq 0$.[1]

Then the conjugate points lie outside the interval $(-1, 1)$.

We have already noted that in the case of the nodes of P. L. Chebyshev the conjugate points lie outside $(-1, 1)$. It is obvious that this assertion is a special case of Theorem 4, when $\alpha^{(n)} = \beta^{(n)} = -\frac{1}{2}$.

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CITED LITERATURE

^{1} S. N. Bernstein, *Collected Works*, 2, Publ. Acad. Sci. USSR, 1954, p. 130.
^{2} D. L. Berman, *DAN*, 60, No. 3 (1948).
^{3} D. L. Berman, *DAN*, 81, No. 1 (1951).
^{4} D. L. Berman, *DAN*, 70, No. 2 (1950).

$$J_n(x, -1, -1) = \int_{-1}^{x} P_{n-1}(t)\, dt,$$

where $P_n(t)$ is the Legendre polynomial of the n-th degree.

Note: Figure translations are in progress. See original paper for figures.

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[1] By definition,