ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SYSTEMS OF LINEAR EQUATIONS

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Abstract

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ON THE ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SYSTEMS OF LINEAR EQUATIONS

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The main result of the present note concerns solutions of a system of the form

$$AY = F,$$

where

$$A = (\alpha_{ij})_1^\infty, \quad Y = \{y_1, y_2, ...\}, \quad F = \{f_1, f_2, ...\},$$

and with respect to the matrix A it is assumed only that its elements are comparatively smooth functions of the indices. We shall precede this result by several others, which are more simply formulated and explain the path by which the main result was obtained.

Let us introduce some notation. Denote by $P(z)$ the polynomial

$$P(z) = \sum_{m=-k_1}^{k_2} a_m z^m, \quad a_{-k_1} \neq 0, \quad a_{k_2} \neq 0,$$

and by $\lambda_{-k_1}, \dots, \lambda_{-1}, \lambda_1, \dots, \lambda_{k_2}$ its zeros, arranged in order of decreasing moduli (for symmetry of notation λ_0 is absent). We shall consider two types of conditions on the zeros:

$$\lambda_i \neq \lambda_j, \quad i \neq j; \quad |\lambda_{-1}| > |\lambda_1|; \tag{1}$$

$$|\lambda_{-k_1}| > \cdots > |\lambda_{-1}| > |\lambda_1| > \cdots > |\lambda_{k_2}|. \tag{2}$$

Denote by A_n the matrix

$$A_n = (\alpha_{ij})_1^n, \quad \alpha_{ij} = \begin{cases} a_{i-j}, & -k_1 \le i-j \le k_2, \\ 0, & i-j < -k_1,\ i-j > k_2, \end{cases}$$

and by $Y^{(n)} = \{y_1^{(n)}, \dots, y_n^{(n)}\}$ the solution of the system

$$A_n Y^{(n)} = e_1, \quad e_1 = \{1, 0, \dots, 0\}. \tag{3}$$

Theorem 1. If $P(z)$ has no multiple zeros, then

$$y_m^{(n)} = -\frac{1}{a_{-k_1}} \frac{V(\lambda_{-k_1}, \dots, \lambda_{k_2};\ m, -k_2+2, -k_2+3, \dots, 0, n+1, \dots, n+k_1)}{V(\lambda_{-k_1}, \dots, \lambda_{k_2};\ -k_2+1, -k_2+2, \dots, 0, n+1, \dots, n+k_1)},$$

where $V(\alpha_1, \dots, \alpha_s;\ n_1, \dots, n_s)$ denotes the generalized Vandermonde determinant

$$V(\alpha_1, \dots, \alpha_s;\ n_1, \dots, n_s) = \det(\alpha_j^m)_1^s.$$

Let us note two corollaries of this theorem.

Corollary 1. If condition (1) on the zeros of $P(z)$ is satisfied, then we have

$$\lim_{n\to\infty} y_m^{(n)} = b_1 \lambda_1^m + \dots + b_{k_2} \lambda_{k_2}^m,$$

where

$$b_s = -\frac{1}{a_{-k_1}} \lambda_s^{-k_2+1} \prod_{\substack{i=1 \\ (i\ne s)}}^{k_2} (\lambda_s - \lambda_i)^{-1}.$$

Corollary 2. If condition (2) is satisfied, then at the zeros of $P(z)$ we have, as $n \to \infty$,

$$\det A_n \sim b \left(\frac{a_{-k_1}}{\lambda_1 \cdots \lambda_{k_2}} \right)^n \sim \frac{b}{2\pi i} \int_{|z|=r} [P(z)]^n \frac{dz}{z},$$

where

$$C = \frac{V(\lambda_{-k_1}, \dots, \lambda_{-1})\, V(\lambda_1, \dots, \lambda_{k_2})}{V(\lambda_{-k_1}, \dots, \lambda_{k_2})}.$$

(Here $V(\alpha_1, \dots, \alpha_s) = \det(\alpha_j^{i-1})_1^s$ is the Vandermonde determinant.)

Theorem 1 and its corollaries play in what follows the same role as equations with constant coefficients in the theory of linear equations.

Let us now have a sequence of polynomials

$$P_s(z) = \sum_{m=-k_1}^{k_2} a_m^{(s)} z^m, \qquad a_{-k_1}^{(s)} \neq 0, \qquad a_{k_2}^{(s)} \neq 0, \qquad s = 1, 2, \dots,$$

with zeros $\lambda_{-k_1}^{(s)}, \dots, \lambda_{-1}^{(s)}, \lambda_1^{(s)}, \dots, \lambda_{k_2}^{(s)}$. Suppose these polynomials satisfy the conditions

$$\sum_{s=1}^{\infty} \left| a_m^{(s+1)} - a_m^{(s)} \right| < \infty, \qquad -k_1 \leq m \leq k_2. \tag{4}$$

In this case $P_s(z)$, $a_m^{(s)}$, $\lambda_m^{(s)}$ tend to limits as $s \to \infty$. Put

$$\lim_{s\to\infty} P_s(z) = P(z), \qquad \lim_{s\to\infty} \lambda_m^{(s)} = \lambda_m, \qquad \lim_{s\to\infty} a_m^{(s)} = a_m, \qquad a_{-k_1} \neq 0, \quad a_{k_2} = 0.$$

Denote by A the matrix

$$A = (\alpha_{ij})_1^{\infty}, \qquad \alpha_{ij} = \begin{cases} a_{i-j}^{(i)}, & -k_1 \leq i-j \leq k_2, \\ 0, & i-j < -k_1,\ i-j > k_2, \end{cases}$$

and by A_n the matrix $A_n = (\alpha_{ij})_1^n$. Finally, let $Y^{(n)} = \{y_1^{(n)}, \dots, y_n^{(n)}\}$ denote the solution of the system

$$A_n Y^{(n)} = F_n, \quad F_n = \{f_1,\ f_2, \dots,\ f_n\}, \qquad f_n = 0, \quad n > n_0.$$

In this notation the following assertion holds.

Theorem 2. If the λ_j satisfy conditions (1), then

$$\lim_{n\to\infty} y_m^{(n)} = b_1 y_{1m} + \dots + b_{k_2} y_{k_2 m},$$

where the b_s depend on F_n, and $y_{sm} \sim \lambda_s^{(1)} \lambda_s^{(2)} \cdots \lambda_s^{(m)}$, $m \to \infty$.

If, moreover, the λ_j satisfy conditions (2), then also

$$\det A_n \sim \frac{b}{2\pi i}\int_{|z|=r} P_1(z)\cdots P_n(z)\,\frac{dz}{z}.$$

Let us now proceed to the formulation of the main result. Suppose we have a sequence of functions

$$P_n(z) = \sum_{m=-n+1}^{\infty} a_m^{(n)} z^m, \qquad 0 < |z| < d_n, \qquad n = 1, 2, \dots$$

We shall assume that the functions $P_n(z)$ satisfy the following four conditions:

1. There exist two sequences ρ_n and R_n, $\rho_n < R_n$, such that

$$\lim_{n\to\infty}\max_{|z|=1}\left|\frac{P_{n+1}\left(\dfrac{z}{R_{n+1}}\right)}{P_n\left(\dfrac{z}{R_n}\right)}\right| = \lim_{n\to\infty}\max_{|z|=1}\left|\frac{P_{n+1}\left(\dfrac{z}{\rho_{n+1}}\right)}{P_n\left(\dfrac{z}{\rho_n}\right)}\right| = 1.$$

2. There exists an r_n, $\rho_n < r_n < R_n$, such that $P_n(z) \neq 0$ for $|z| = r_n$ and

$$\operatorname*{var}_{|z|=r_n} \arg P_n(z) = 0.$$

3. For sufficiently large n, the function $P_n(z)$ has in the ring $r_n \le |z| \le R_n$ exactly k_1 zeros, say $\lambda_{-1}^{(n)}, \dots, \lambda_{-k_1}^{(n)}$, and in the ring $\rho_n \le |z| \le r_n$ exactly k_2 zeros, say $\lambda_1^{(n)}, \dots, \lambda_{k_2}^{(n)}$, with $\lambda_i \neq \lambda_j$ for $i \neq j$.

4. Put $P_{n,m}(z) = \dfrac{P_n(z)}{z - \lambda_m^{(n)}}$. Let

$$\lim_{n\to\infty}\frac{P_{n+1,i}(\lambda_i^{(n)})}{P_{n+1,i}(\lambda_i^{(n+1)})} = 1, \qquad \sum_{n=1}^{\infty}\left|\beta_{ij}(n+1) - \beta_{ij}(n)\right| < \infty,$$

$$\sum_{n=1}^{\infty}|\gamma_{ij}(n)| < \infty, \qquad i \neq j;$$

$$\beta_{ij}(n) = \frac{P_{n+1,j}(\lambda_j^{(n)})}{\lambda_j^{(n)} P_{n+1,i}(\lambda_j^{(n)})}, \qquad \gamma_{ij}(n) = \lambda_j(n)\beta_{ij}(n)\beta_{ji}(n).$$

Denote by A the matrix

$$A = (\alpha_{ij})_1^{\infty}, \qquad \alpha_{ij} = a_{j-i}^{(i)},$$

and by $Y = \{y_1, y_2, ...\}$ a solution of the system

$$AY = F, \qquad F = \{f_1, f_2, ...\}, \qquad f_n = 0, \quad n > n_1. \tag{5}$$

Concerning the system (5), we shall assume, in addition, that it has a solution for every right-hand side of the indicated form.

Under the enumerated conditions the following assertion holds.

Theorem 3. Let $\varepsilon > 0$ be arbitrarily small. Any solution of the system (5) satisfying the condition

$$y_n = O((1-\varepsilon)^n R_1 \cdots R_n)$$

has the form

$$y_n = C_{-1} y_{-1,n} + \cdots + C_{-k_1} y_{-k_1,n} + b_1 y_{1,n} + \cdots + b_{k_2} y_{k_2,n} + O((1+\varepsilon)^n \rho_1 \cdots \rho_n),$$

where $C_{-1}, \ldots, C_{-k_1}$ are arbitrary constants; $b_1, \ldots, b_{k_2}$ are constants determined by F;

$$y_{m,n} \sim \mu_m^{(1)} \mu_m^{(2)} \cdots \mu_m^{(n)}, \qquad \mu_m^{(n)} = \lambda_m^{(n)} \frac{P_{n+1,m}(\lambda_m^{(n)})}{P_{n+1,m}(\lambda_m^{(n+1)})}.$$

Let us also note the continuous analogue of this theorem. Consider the integral equation

$$y(x) + \int_0^\infty K(x, x-t) y(t)\, dt = f(x), \qquad x \geq 0, \tag{6}$$

whose kernel satisfies the following four conditions:

1. There exist two functions $\rho(x)$ and $R(x)$, $\rho(x) < R(x)$, such that the integral entering the formula

$$P(x,z) = 1 + \int_{-x}^\infty K(x,t) e^{-tz}\, dt,$$

converges in the strip $\rho(x) \leqslant \operatorname{Re} z \leqslant R(x)$, and

$$\lim_{x\to\infty} \max_{\operatorname{Re} z = 1} \left| \frac{d}{dx} \ln P\left(x, \frac{z}{R(x)}\right) \right| = \lim_{x\to\infty} \max_{\operatorname{Re} z = 1} \left| \frac{d}{dx} \ln P\left(x, \frac{z}{\rho(x)}\right) \right| = 0.$$

2. There exists a function $r(x)$, $\rho(x) < r(x) < R(x)$, such that $P(x,z) \neq 0$ for $\operatorname{Re} z = r(x)$, and

$$\operatorname*{var}_{\operatorname{Re} z = r(x)} \arg P(x,z) = 0.$$

3. For sufficiently large x, the function $P(x,z)$ has in the strip $r(x) \leqslant \operatorname{Re} z \leqslant R(x)$ exactly k_1 zeros, say $\lambda_{-1}(x), \dots, \lambda_{-k_1}(x)$, and in the strip $\rho(x) \leqslant \operatorname{Re} z \leqslant r(x)$ exactly k_2 zeros, say $\lambda_1(x), \dots, \lambda_{k_2}(x)$, with $\lambda_i(x) \neq \lambda_j(x)$ for $i \neq j$.

4. Put $P_m(x,z) = \dfrac{P(x,z)}{z - \lambda_m(x)}$. Let

$$\int_0^\infty |\beta'_{ij}(x)|\,dx < \infty, \qquad \int_0^\infty |\gamma_{ij}(x)|\,dx < \infty, \qquad i \neq j;$$

$$\beta_{ij}(x) = \frac{\lambda'_j(x)}{\lambda_j(x)} - \frac{P'_{iz}(x, \lambda_j(x))}{P_i(x, \lambda_i(x))}, \qquad \gamma_{ij}(x) = \lambda_j(x)\beta_{ij}(x)\beta_{ji}(x).$$

In addition, suppose that $f(x)$ is different from zero only on a finite interval and that, for every such $f(x)$, the integral equation has a solution.

Under these assumptions the following assertion holds.

Theorem 4. Let $\varepsilon > 0$ be arbitrarily small. Every solution of equation (6) satisfying the condition

$$y(x) = O\left(\exp\left\{(1-\varepsilon)\int_0^x R(t)\,dt\right\}\right)$$

has the form

$$y(x) = C_{-1}y_{-1}(x) + \dots + C_{-k_1}y_{-k_1}(x) + b_1 y_1(x) + \dots + b_{k_2}y_{k_2}(x) +$$

$$+ O\left(\exp\left\{(1+\varepsilon)\int_0^\infty \rho(t)\,dt\right\}\right),$$

where $C_{-1}, \dots, C_{-k_1}$ are arbitrary constants; $b_1, \dots, b_{k_2}$ are constants determined by $f(x)$; $y_m(x) \sim \exp\left\{\int_0^x \mu_m(t)\,dt\right\}$, $\mu_m(x) =$

$$= \lambda_m(x) - \lambda'_m(x)\frac{P'_{mz}(x, \lambda_m(x))}{P_m(x, \lambda_m(x))}.$$

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