MATHEMATICS

Abstract

Full Text

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EMBEDDING THEOREMS FOR FUNCTIONS WITH PARTIAL DERIVATIVES CONSIDERED IN DIFFERENT METRICS

(Presented by Academician S. L. Sobolev on 25 VI 1957)

In our papers (see, in particular, ([5,6])) we obtained a generalization of the theorem of S. L. Sobolev ([8,9]) and the theorem of V. I. Kondrashov ([4,9]) on the embedding of classes of differentiable functions of many variables. From this point of view we considered classes of functions $H_p^{(r_1,\dots,r_n)}$, where the positive numbers r_i indicate the differential properties of the function $f \in H_p^{(r_1,\dots,r_n)}$ with respect to each variable x_i separately $(i = 1, \dots, n)$, and the number p indicates the metric L_p by means of which the function f and its partial derivatives are normed.

However, one may introduce more general classes $H_{p_1,\dots,p_n}^{(r_1,\dots,r_n)}$ of functions f, where the numbers p_i, generally speaking different, indicate that the partial derivatives (not mixed) of the function f with respect to each variable x_i are considered in their own metric L_{p_i} $(i = 1, \dots, n)$. For these classes we have succeeded in obtaining the corresponding embedding theorems, which generalize our previous theorems.

We shall consider real-valued functions $f = f(x_1, \dots, x_n)$ defined on the n-dimensional space R_n. Put

$$\|f\|_{L_p^{(m)}} = \left(\int_{R_m} |f(x_1, \dots, x_m, x_{m+1}, \dots, x_n)|^p \, dx_1 \dots dx_m\right)^{1/p} \qquad (m = 1, \dots, n).$$

Thus $\|f\|_{L_p^{(m)}}$ for $m < n$ depends on $x_{m+1}, \dots, x_n$.

Let positive numbers M, r_i, and p_i be given, where $1 \le p_i \le \infty$. Thus, in particular, the p_i may be equal to $+\infty$. By definition, if a function f belongs simultaneously to the classes* $H_{p_i x_i}^{(r_i)}(M)$, then we shall say that it belongs to the class $H_{p_1,\dots,p_n}^{(r_1,\dots,r_n)}(M)$. If $p_1 = \dots = p_n = p$, then we put $H_{p_1,\dots,p_n}^{(r_1,\dots,r_n)} = H_p^{(r_1,\dots,r_n)}(M)$.

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* A function $f \in H^{(r)}_{px_1}(M)$ (see ([5]) or ([6])) if it is integrable to the p-th power on R_n together with its partial derivatives $\partial^k f/\partial x_1^k$ $(k = 0, 1, \dots, \bar{r})$, where $r = \bar{r} + \alpha$, $\bar{r}$ is an integer and $0 < \alpha \le 1$, and, moreover,

$$\left\| f^{(\bar{r})}_{x_1}(x_1 + h, x_2, \dots, x_n) - f^{(\bar{r})}_{x_1}(x_1, \dots, x_n) \right\|_{L_p^{(n)}} \le M|h|^\alpha, \quad \text{if } \alpha < 1;$$

$$\left\| f^{(\bar{r})}_{x_1}(x_1 + h, x_2, \dots, x_n) - 2f^{(\bar{r})}_{x_1}(x_1, \dots, x_n) + f^{(\bar{r})}_{x_1}(x_1 - h, x_2, \dots, x_n) \right\|_{L_p^{(n)}} \le M|h|, \quad \text{if } \alpha = 1.$$

Theorem 1. Let, for the numbers considered below, the inequalities
$r_i > 0; \quad 1 \le p_i \le q \le \infty; \quad n,\ m$ be natural numbers,
$1 \le m \le n$, hold,

$$\rho_i = \frac{r_i \chi}{\chi_i} > 0 \qquad (i = 1, \dots, n), \tag{1}$$

where

$$\chi = \begin{vmatrix} 1 - \sum\limits_1^n \dfrac{\frac{1}{p_l} - \frac{1}{q}}{r_l} & -\dfrac{1}{q} \sum\limits_1^n \dfrac{1}{r_l} \\ -\sum\limits_{m+1}^n \dfrac{\frac{1}{p_l} - \frac{1}{q}}{r_l} & 1 - \dfrac{1}{q} \sum\limits_{m+1}^n \dfrac{1}{r_l} \end{vmatrix}, \qquad \chi_i = 1 - \sum_{l=1}^n \frac{\frac{1}{p_l} - \frac{1}{p_i}}{r_l}.$$

Let, moreover, a function defined in the n-dimensional space R_n satisfy
$f \in H^{(r_1, \dots, r_n)}_{p_1, \dots, p_n}(M)$. Then for any fixed $(x_{m+1}, \dots, x_n)$, f, as a function of $x_1, \dots, x_m$, belongs to the class $H_q^{(\rho_1, \dots, \rho_m)}(\overline{M})$. In this case the inequality

$$\|f\|_{L_q^{(m)}} + \overline{M} < c \left(\min_i \|f\|_{L_{p_i}^{(n)}} + M \right), \tag{2}$$

holds, where the constant c does not depend on the set standing in the series.

If not only $\chi > 0$, but also* $1 - \sum_1^n \frac{1}{p_k r_k} > 0$, then for any $\varepsilon > 0$ there exists a function
$f \in H^{(r_1, \dots, r_n)}_{p_1, \dots, p_n}(M)$, which, for fixed $x_{m+1}, \dots, x_n$, as a function of $x_1, \dots, x_m$, does not belong to the class
$H_q^{(\rho_1, \dots, \rho_{i-1}, \rho_i - \varepsilon, \rho_{i+1}, \dots, \rho_n)}(M_1)$ for any M_1.

This theorem is proved on the basis of approximating functions of the named classes by entire functions

$$g = g_{\nu_1, \dots, \nu_n} = g_{\nu_1, \dots, \nu_n}(x_1, \dots, x_n) \tag{3}$$

of degrees $\nu_1, \dots, \nu_n$ respectively in the variables $x_1, \dots, x_n$. In doing so, the following theorem proved useful; it is a generalization of Jackson' s theorem.

Theorem 2. For any positive r_i and p_i there exists a constant c, depending on these numbers, such that, whatever the function
$f \in H^{(r_1,\dots,r_n)}_{p_1,\dots,p_n}(M)$, there can be found for it a system of functions (3) $(1 \le \nu_i \le \infty,\ i = 1, \dots, n)$ for which the inequalities** hold:

$$\left\| f - g_{\nu_1,\infty,\dots,\infty} \right\|_{L^{(n)}_{p_1}} < \frac{cM}{\nu_1^{r_1}},$$

$$\left\| g_{\nu_1,\infty,\dots,\infty} - g_{\nu_1,\nu_2,\infty,\dots,\infty} \right\|_{L^{(n)}_{p_2}} < \frac{cM}{\nu_2^{r_2}}, \tag{4}$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\left\| g_{\nu_1,\dots,\nu_{n-1},\infty} - g_{\nu_1,\dots,\nu_n} \right\|_{L^{(n)}_{p_n}} < \frac{cM}{\nu_n^{r_n}}.$$

* We do not think that the restriction $1 - \sum_1^n \frac{1}{p_k r_k} > 0$ is essential. Without it, one should suppose, the corresponding example can be obtained analogously to how, in the case $p_1 = \dots = p_n = p$, T. I. Amanov reasoned ([1]).

** If $\nu_k = \infty$, this means that g need not be an entire function with respect to x_k.

For $p = p_1 = \dots = p_n$, Theorem 2 contains our earlier result (([5], p. 267), which for $p = \infty$ turns into a result of S. N. Bernstein ([2]).

Remark 1. Theorem 1 remains valid without changes for the class $H^{(r_1,\dots,r_n)*}_{p_1,\dots,p_n}$ of functions f, periodic in each of the variables x_i, defined similarly to the class $H^{(r_1,\dots,r_n)}_{p_1,\dots,p_n}$, but where the integrals serving to define the norm $\|f\|_{L^{(m)}_p}$ are calculated not over the whole space, but over a period. There is also a corresponding analogue of Theorem 2, where the role of $P_{\nu_1,\dots,\nu_n}$ is played by trigonometric polynomials.

Remark 2. Let us note cases when Theorem 1 retains its force for classes of functions $f \in H^{(r_1,\dots,r_n)}_{p_1,\dots,p_n}(G; M)$, given on a domain $G \subset R_n$. These classes are defined analogously to $H^{(r_1,\dots,r_n)}_{p_1,\dots,p_n}$.

a) If G is a rectangular parallelepiped with faces parallel to the coordinate axes, then a function $f \in H^{(r_1,\dots,r_n)}_{p_1,\dots,p_n}(G; M)$ can be extended to R_n so that the extended function $\bar{f} \in H^{(r_1,\dots,r_n)}_{p_1,\dots,p_n}$, in any case if the r_i are not integers. For integer r_i, this is probably also true, but has not yet been proved.

b) On the other hand, if $r_1, \dots, r_m$ are integers and $r_{m+1}, \dots, r_n$ are not integers, if by the class $H^{(r_1,\dots,r_n)}_{p_1,\dots,p_n}(G; M)$ one understands the intersection of the classes* $W^{(r_i)}_{p_i x_i}(G; M)$ $(i = 1, \dots, m)$ and $H^{(r_i)}_{p_i x_i}(G; M)$ $(i = m+1, \dots, n)$, then a function of this class can certainly be extended, preserving these properties, to R_n beyond the parallelepiped G defined above, and then the extended function $\bar{f} \in H^{(r_1,\dots,r_n)}_{p_1,\dots,p_n}(cM)$.

c) Let $G = G' \times G''$ be the topological product of domains $G' \subset R_l(x_1, \dots, x_l)$ and $G'' \subset R_{n-l}(x_{l+1}, \dots, x_n)$, bounded by surfaces continuously differentiable $r + 2$ times, and let the function f, given on G, have partial derivatives

$$\frac{\partial^{r'} f}{\partial x_1^{\alpha_1} \cdots \partial x_l^{\alpha_l}} \left(\sum_{k=1}^{l} \alpha_k \le r' \right),$$

belonging to $L_{p'}(G)$, as well as partial derivatives

$$\frac{\partial^{r''} f}{\partial x_{l+1}^{\beta_{l+1}} \cdots \partial x_n^{\beta_n}} \left(\sum_{l+1}^{n} \beta_k \le r'' \right),$$

belonging to $L_{p''}(G)$. Then f can be extended beyond G to R_n with preservation of the indicated differentiability properties. Thus the extended function $\bar{f} \in H^{(r_1,\dots,r_n)}_{p_1,\dots,p_n}(cM)$, where $r_1 = \dots = r_l = r'$, $r_{l+1} = \dots = r_n = r''$, $p_1 = \dots = p_l = p'$, $p_{l+1} = \dots = p_n = p''$.

In conclusion we note that, on the basis of Theorem 1, a corresponding compactness theorem can be obtained, analogous to our results published in ([7]). This generalizes, for the domains considered by us, the compactness theorems obtained by E. Gagliardo ([3]).

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* $f \in W^{(r)}_{px_1}(G; M)$, if $\partial^k f / \partial x^k \in L_p(G)$ $(k = 0, 1, \dots, r)$.

Note: Figure translations are in progress. See original paper for figures.