ON A TRANSFORMATION OF ISOMETRIC SURFACES

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Abstract

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ON A TRANSFORMATION OF ISOMETRIC SURFACES

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In this note a standard method will be given for assigning to each pair of isometric surfaces of a space of constant curvature $K \neq 0$ a pair of isometric surfaces of Euclidean space, and conversely, to each pair of isometric surfaces of Euclidean space–a pair of isometric surfaces of a space of constant curvature. This gives a new approach to the problem of unique determination of convex surfaces in spaces of constant curvature.

Without loss of generality, we shall assume $K = 1$ for elliptic space and $K = -1$ for Lobachevsky space.

Let R be elliptic space. Introduce Weierstrass coordinates x_i $(i = 0, 1, 2, 3)$ in R, and associate with each point of the space R a pair of points of four-dimensional Euclidean space with Cartesian coordinates x_i and $-x_i$. These points fill the unit sphere, since the Weierstrass coordinates satisfy the condition

$$x^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1.$$

Denote by E_0 the three-dimensional Euclidean space $x_0 = 0$.

Theorem 1. *Suppose that in elliptic space R we have two isometric surfaces F' and F'', given by equations in Weierstrass coordinates*

$$x = x'(u, v), \qquad x = x''(u, v),$$

where the points of the surfaces corresponding under the isometry have the same coordinates u, v.

Then the equations in Cartesian coordinates y_i

$$y = \frac{x'(u,v) - e_0(x'(u,v)e_0)}{e_0(x'(u,v) \pm x''(u,v))},$$

$$y = \frac{x''(u,v) - e_0(x''(u,v)e_0)}{e_0(x'(u,v) \pm x''(u,v))} \tag{1}$$

define two isometric surfaces in the Euclidean space E_0.

In equations (1), e_0 is the unit vector along the axis x_0, and the scalar product is expressed by the usual formula.

Let now R be Lobachevsky space ($K = -1$). Introduce Weierstrass coordinates x_i in R, and associate with each point of R the point of four-dimensional Euclidean space with Cartesian coordinates x_i. Then R is mapped onto the two-sheeted hyperboloid

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1.$$

Theorem 2. *Suppose that in Lobachevsky space R we have two isometric surfaces F' and F'', given by equations in Weierstrass coordinates*

$$x = x'(u,v), \qquad x = x''(u,v).$$

Then the equations in Cartesian coordinates y_i

$$y = \frac{x'(u,v) + e_0(x'(u,v)e_0)}{e_0(x'(u,v) \pm x'(u,v))},$$

$$y = \frac{x''(u,v) + e_0(x''(u,v)e_0)}{e_0(x'(u,v) \pm x''(u,v))} \tag{2}$$

define two isometric surfaces in the Euclidean space E_0.

In equations (2) the scalar product is taken with respect to the form

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

Theorem 3. Let in the Euclidean space E_0 there be two isometric surfaces Φ' and Φ'', given in Cartesian coordinates y_i by the equations

$$y = y'(u,v), \qquad y = y''(u,v).$$

Then the equations in Weierstrass coordinates x_i

$$\begin{aligned} x &= \rho\{2y'(u,v) \pm e_0(1 - y''^2(u,v) + y''^2(u,v))\}, \\ x &= \rho\{2y''(u,v) \pm e_0(1 - y''^2(u,v) + {y'}^2(u,v))\} \end{aligned} \tag{3}$$

define two isometric surfaces in the elliptic space R ($K = 1$).

In equations (3) ρ is a normalizing factor, determined by the condition $x^2 = 1$, and the scalar product is determined by the usual formula for vectors of four-dimensional space.

Theorem 4. Let in the Euclidean space E_0 there be two isometric surfaces Φ' and Φ'', given in Cartesian coordinates y_i by the equations

$$y = y'(u, v), \qquad y = y''(u, v).$$

Then the equations in Weierstrass coordinates x_i

$$x = \rho\{2y'(u, v) \pm e_0(1 - y''^2(u, v) + {y'}^2(u, v))\},$$

$$x = \rho\{2y''(u, v) \pm e_0(1 - {y'}^2(u, v) + y''^2(u, v))\} \tag{4}$$

define two isometric surfaces in Lobachevsky space R $(K = -1)$.

In equations (4) the scalar multiplication is taken with respect to the quadratic form

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2,$$

and the normalizing factor ρ is determined by the condition $x^2 = -1$.

In all Theorems 1-4, if the given surfaces are simply congruent, i.e. can be brought into coincidence by a motion, then the surfaces obtained are also congruent.

We shall give an example of the use of Theorem 1 for proving the unique determination of closed convex surfaces in elliptic space.

Let F' and F'' be closed isometric, equally oriented convex surfaces in elliptic space R $(K = 1)$. Place the surfaces F' and F'' so that they do not intersect the plane $x_0 = 0$. Then the Weierstrass coordinates of the points of the surfaces can be normalized by the additional condition $x_0 > 0$. Move the given surfaces F' and F'', remaining in the region $x_0 > 0$, into such a position that the point $(1, 0, 0, 0)$ lies inside each of the surfaces.

If we now take equations (1) with the plus sign in the denominator, then the corresponding surfaces of the Euclidean space E_0 will be not only isometric but also convex. And such surfaces, as is known, are congruent. Consequently, the surfaces F' and F'' are also congruent.

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Note: Figure translations are in progress. See original paper for figures.