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Abstract

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ON THE APPROXIMATE SOLUTION OF THE QUANTUM MANY-BODY PROBLEM IN THE CASE OF BOSE STATISTICS

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In ([1]) it was shown that an increase in the density and in the interaction energy of the particles of a Bose gas leads to a decrease in the number of particles in the condensate N_0. It is therefore of interest to investigate the case for which N_0 is small. In view of this, in the Hamiltonian of the system

$$H = T + U = \sum T_k a_k^+ a_k + \frac{1}{2} \sum V_q a_k^+ a_l^+ a_{k+q} a_{l-q}$$

we shall separate out the terms with a_0^+ and a_0, but shall not regard them as large. We shall first assume that $T \sim U$ and shall try, along with T, to take into account the principal terms in U. Since for a rarefied Bose gas the leading role is played by the interaction of particles with oppositely directed momenta ([1,2]), in our case as well we shall apply a canonical transformation to new amplitudes $\alpha_k,\ \alpha_k^+$ of the same type as in ([1]): $a_k = u_k \alpha_k + v_k \alpha_{-k}^+;\ u_k^2 - v_k^2 = 1$. The parameters $u_k,\ v_k$ will be found from the requirement that the mean value of H be minimal with respect to the new occupation numbers, with the additional condition that the number of particles be constant:

$$\widetilde{E} = H_{\text{av}} - \mu N = \frac{1}{2} V_0 N^2 - \mu N_0 + \sum (T_k + N_0 V_k - \mu) \left[n_k + (2n_k + 1) v_k^2 \right] +$$

$$+ N_0 \sum V_k u_k v_k (2n_k + 1) + \frac{1}{2} \sum V_{|l-k|} \{ u_k v_k u_l v_l (2n_k + 1)(2n_l + 1) +$$

$$+ \left[n_k + (2n_k + 1) v_k^2 \right] \left[n_l + (2n_l + 1) v_l^2 \right] \}.$$

Here small terms not proportional to the volume Ω (as $\Omega \to \infty$) have been discarded; in the summations here and below $k \neq 0;\ l \neq 0;\ n_k = \alpha_k^+ \alpha_k;\ N =$

$N_{\rm av} = N_0 + \sum[n_k + (2n_k+1)v_k^2]$; μ is a Lagrange multiplier, which coincides with the chemical potential.

Minimizing $\widetilde{E}$ with respect to $u_k,\ v_k$, we find

$$2u_k v_k \xi_k = -(u_k^2 + v_k^2) g_k,$$

where

$$\xi_k = T_k + N_0 V_k - \mu + \sum V_{|l-k|}\left[n_l + (2n_l+1)v_l^2\right];$$

$$g_k = N_0 V_k + \sum V_{|l-k|} u_l v_l (2n_l + 1).$$

Hence we obtain

$$u_k^2 = \frac{1}{2}\left(\frac{\xi_k}{\varepsilon_k} + 1\right); \qquad v_k^2 = \frac{1}{2}\left(\frac{\xi_k}{\varepsilon_k} - 1\right),$$

where

$$\varepsilon_k = (\xi_k^2 - g_k^2)^{1/2}.$$

The excitation energy $\delta\widetilde{E}/\delta n_k$ proves to be equal to ε_k.

Minimizing the free energy of the system (see (3)), we find $n_k = [\exp(\varepsilon_k/\theta) - 1]^{-1}$. Substituting the expressions for u_k, v_k into the expressions for g_k, ξ_k, we obtain integral equations for determining g_k, ξ_k:

$$g_k = N_0 V_k - \sum V_{|1-k|}(n_l + 1/2) g_l (\xi_l^2 - g_l^2)^{-1/2},$$

$$\xi_k = T_k + N_0 V_k - \mu + \sum V_{|1-k|}\{n_l + (n_l + 1/2)[\xi_l(\xi_l^2 - g_l^2)^{-1/2} - 1]\}.$$

We find μ from the condition for N. As $\theta \to 0$, for all $\varepsilon_k > 0$, $n_k \to 0$, and since $\sum v_k^2 \ll N$ (see below), in order to satisfy the condition $N = \text{const}$ it is necessary that $n(\varepsilon_{\min}) = n_0 \to \infty$ ($N_0 \to \infty$) as $\Omega \to \infty$ (i.e. $n_0 \sim \Omega$ for finite Ω). Consequently, $\varepsilon_0 = 0$, i.e. $|\xi_0| = |g_0|$. It is not difficult to see that in this case $\varepsilon_k \to sk$ as $k \to 0$, i.e. the excitations are sound-like.

First let us solve the equations for g_k and ξ_k for the simple case when $N - N_0 \ll N$. Let $g_k = g_{k0} + g_k'$; $\xi_k = \xi_{k0} + \xi_k'$; $\mu = \mu_0 + \mu'$, where $g_{k0} = N_0 V_k$, $\xi_{k0} = T_k + N_0 V_k$, $\mu_0 = 0$. Then, obviously,

$$\mu' = \mu = \sum V_k \left\{ n_{k0} + (n_{k0} + 1/2) \left[\frac{\xi_{k0} + g_{k0}}{\varepsilon_{k0}} - 1 \right] \right\};$$

$$g'_k = -N_0 \sum V_l V_{|1-k|} \frac{n_{k0} + 1/2}{\varepsilon_{l0}};$$

$$\xi'_k = \sum (V_{|1-k|} - V_l) \left\{ n_{l0} + (n_{l0} + 1/2) \left[\frac{\xi_{l0}}{\varepsilon_{l0}} - 1 \right] \right\} - N_0 \sum V_l^2 \frac{n_{l0} + 1/2}{\varepsilon_{l0}};$$

Here $n_{l0} = n_l(\varepsilon_{l0})$; $\varepsilon_{l0} = [(2N_0 V_l + T_l)T_l]^{1/2}$; N_0 is a function of θ;

$$N_0 = N - \sum \left[1/2 \left(\frac{\xi_k}{\varepsilon_k} - 1 \right) + n_k \frac{\xi_k}{\varepsilon_k} \right].$$

Of considerably greater interest is the case when $N_0 \lesssim N$, in particular $N_0 \ll N$. From the condition $\xi_0 = g_0$ we find:

$$\mu = \sum V_k \{ n_k + (n_k + 1/2)[(\xi_k + g_k)(\xi_k^2 - g_k^2)^{-1/2} - 1] \}.$$

For g_k, ξ_k we obtain the equations

$$g_k = N_0 V_k - \sum V_{|1-k|} (n_l + 1/2) g_l (\xi_l^2 - g_l^2)^{-1/2};$$

$$\xi_k = T_k + N_0 V_k + \sum (V_{|1-k|} - V_l) \{ n_l + (n_l + 1/2)[\xi_l (\xi_l^2 - g_l^2)^{-1/2} - 1] \} - \sum V_l (n_l + 1/2) g_l (\xi_l^2 - g_l^2)^{-1/2}.$$

The solution of these equations will be considered in another paper. It is not difficult to see that in this case, for $\theta \neq 0$, a negative sound dispersion is also possible. The expression written above for μ, obviously, is valid not only for $\theta = 0$, but also for $\theta \neq 0$, if a condensate exists ($N_0 \sim \Omega$), i.e. in the superfluid phase. Using the formula $\mu = \partial E / \partial N$, one can find an expression for μ also in the normal phase. The case $N_0 \ll N$ makes it possible to consider the transition of the superfluid phase into the normal one. The phase-transition temperature θ_k is determined by the condition

$$N_0 = N - \frac{\Omega}{(2\pi)^3} \int \left[1/2 \left(\frac{\xi_k}{\varepsilon_k} - 1 \right) + n_k \frac{\xi_k}{\varepsilon_k} \right] dk = 0.$$

The jump in the heat capacity is connected with the vanishing of N_0 and the change in the form of ε_k.

Let us now consider the condition of applicability of our theory. If V_q contains a small dimensionless parameter g, then, in addition to the requirement that g be small, it is also necessary to require that the correction to the Ψ-function of the system,

associated with the terms in T that are nondiagonal in α_k, α_k^+, be small. These terms have the form

$$T_{\text{n.d.}} = \sum u_k v_k T_k (\alpha_k^+ \alpha_{-k}^+ + \alpha_k \alpha_{-k}).$$

The condition of applicability, obviously, has the form $\sum v_k^2 \ll N$. For $N - N_0 \ll N$, i.e., for $\theta \sim 0$, this condition coincides with that found in (1):

$$\left(\frac{N}{\Omega}\right)^{1/3} \ll \frac{\hbar^2}{m\Omega V_0}.$$

But now the case $N_0 \ll N$, i.e., $\theta \sim \theta_k$, is also possible. For $\theta \neq 0$ the condition for the density N/Ω may turn out to be weaker.

The approximation can be somewhat improved if, instead of the transformation $a_k = u_k \alpha_k + v_k \alpha_{-k}^+$, one uses the more complicated transformation: $a_k = u_k \alpha_k + v_k \alpha_{-k}^+ + \tilde{u}_k \alpha_{-k} + \tilde{v}_k \alpha_{-k}^+$; $(u_k^2 - v_k^2 + \tilde{u}_k^2 - \tilde{v}_k^2 = 1;\ u_k \tilde{u}_k = \tilde{v}_k v_k)$. Here $\tilde{u}_k, \tilde{v}_k$ should be regarded as small quantities compared with u_k, v_k. The corresponding transformation can easily be written also in the case of Fermi statistics. A further approximation can be obtained by using the still more general transformation $a_k = \sum_l (u_{kl} \alpha_l + v_{kl} \alpha_{-l}^+)$, where $v_{kk} \gg v_{kl}$, $u_{kk} \gg u_{kl}$ for $k \neq l$. Terms in H containing u_{kl}, v_{kl} with $k \neq l$ may be treated as a perturbation.

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Note: Figure translations are in progress. See original paper for figures.