# THE THERMAL BOUNDARY LAYER ON A ROTATING DISK

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## Abstract

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## HYDROMECHANICS

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# THE THERMAL BOUNDARY LAYER ON A ROTATING DISK

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1. Let us compare the equation of tangential velocities of the mean turbulent motion of an incompressible fluid in the boundary layer of a rotating disk,

$$v_r \frac{\partial v_\varphi}{\partial r} + \frac{v_r v_\varphi}{r} + v_z \frac{\partial v_\varphi}{\partial z} = \frac{\partial}{\partial z}\left(\nu \frac{\partial v_\varphi}{\partial z} - \overline{v'_z v'_\varphi}\right) \tag{1}$$

with the energy equation without dissipative terms,

$$v_r \frac{\partial T}{\partial r} + v_z \frac{\partial T}{\partial z} = \frac{\partial}{\partial z}\left(a \frac{\partial T}{\partial z} - \overline{v'_z T'}\right), \tag{2}$$

where $v_r, v_\varphi, v_z, T$ are the mean values of the velocity components and temperature, and the primed quantities are the pulsation components (the overbar denotes averaging); $\nu$ is the kinematic viscosity; $a = \lambda / c_p \rho$ is the thermal diffusivity. We note that, for a quadratic temperature distribution on the wall $T_{\text{st}} = r^2$ and $\text{Pr} = \nu / a = 1$ ($T_\infty = 0$), the temperature profiles $\theta(z, r) = T / r^2$ and tangential-velocity profiles $G(z, r) = v_\varphi / r\omega$ coincide, since equations (1) and (2) assume the same form,

$$v_r \frac{\partial R}{\partial r} + 2R \frac{v_r}{r} + v_z \frac{\partial R}{\partial z} = \frac{\partial}{\partial z}\left(\nu \frac{\partial R}{\partial z} - \overline{v'_z R'}\right); \qquad R = G,\ \theta$$

and the boundary conditions are identical,

$$G(0, r) = \theta(0, r) = 1; \qquad G(\infty, r) = \theta(\infty, r) = 0.$$

In doing so we assume that the relations between the pulsation and mean components are the same for $v_\varphi$ and $T$.

Hence we obtain the following relation between the shear stress

$$\tau_\varphi = \mu\, \partial v_\varphi / \partial z - \rho \overline{v'_z v'_\varphi}$$

and the heat flux

$$q = \lambda\,\partial T/\partial z - \rho c_p \overline{v_z' T'}\ :$$

$$q = c_p \tau_\varphi \frac{T_{\text{st}}}{r\omega}. \tag{3}$$

We note that the result obtained also includes the case of laminar flow ([1]), when there are no pulsation components. Using Cochran' s solution for $\tau_\varphi$ ([2]), we obtain in this case

$$\text{Nu} = \frac{qr}{\lambda T_{\text{st}}} = 0.616\,\text{Re}^{1/2}, \qquad \text{Re} = \frac{r^2\omega}{\nu}. \tag{4}$$

For the turbulent regime, according to the refined solution ([3]) and in agreement with experimental data, the moment of resistance $M$ of one side of the disk may be expressed, for $\text{Re} = 10^5 \div 10^7$, in the form

$$M = 0.157\ \text{Re}^{-0.2}\ \frac{1}{4}\rho\omega^2 r^5,$$

which gives

$$\text{Nu} = \frac{dM/dr}{2\pi\omega r^2\mu} = 0.0287\ \text{Re}^{0.8}\,. \tag{5}$$

In the presence of blowing of the disk by an axisymmetric flow perpendicular to the disk surface, the additional velocity components

$$v_r^0 = \alpha r, \qquad v_\varphi^0 = 0, \qquad v_z^0 = -2\alpha z$$

do not change the boundary conditions for $G$; therefore (3) is retained, i.e.,

$$\frac{\text{Nu}}{\text{Nu}\big|_{\alpha=0}} = \frac{\tau_\varphi}{\tau_\varphi\big|_{\alpha=0}} = \Phi\left(\frac{\alpha}{\omega}\right). \tag{6}$$

The function $\Phi(\alpha/\omega)$ is determined in works ([1,4]).

2. To calculate heat transfer for an arbitrary temperature gradient, we set up the heat balance for an annular element of the thermal boundary layer of thickness $\delta_T$ and width $dr$ at a given radius $r$. The secondary heat flux in the radial direction through this element,

$$c_p \rho d \left( \int_0^{\delta_T} 2\pi r v_r T \, dz \right),$$

is balanced by the heat transfer from the disk surface in an annular strip of area $2\pi r \, dr \ / \ 2\pi qr \, dr$.

Hence we obtain the integral relation

$$c_p \rho \frac{d}{dr} \left( \int_0^{\delta_T} r v_r T \, dz \right) = rq. \tag{7}$$

Here the dissipative terms are absent, and also, as in (2), the term associated with thermal conductivity in the radial direction is negligibly small for large Reynolds numbers.

Introducing the conditional thickness of loss of heat content

$$\delta_T^{**} = \int_u^{\delta_T} v_r T \, dz / r\omega T_{\text{st}},$$

we can represent (7) in the form

$$\frac{d\delta_T^{**}}{dr} + \delta_T^{**} \eta = \frac{\text{Nu}}{\text{Pr}\,\text{Re}}, \qquad \eta = \frac{dT_{\text{st}}/dr}{T_{\text{st}}} + \frac{2}{r}; \tag{8}$$

$$\frac{d}{dr} \left( \text{Re}_T^{**} \, r T_{\text{st}} \right) = \frac{\text{Nu} \; T_{\text{st}}}{\text{Pr}}, \qquad \text{Re}_T^{**} = \frac{\delta_T^{**} r \omega}{\nu}. \tag{9}$$

We shall now apply, by analogy with the dynamic boundary layer, the one-parameter method of L. G. Loitsyansky ([5]).

We assume that the form parameter $f_T$ can be represented as

$$f_T = \eta \delta_T^{**} G \left( \text{Re}_T^{**} \right), \tag{10}$$

where $G(\text{Re}_T^{**})$ is a universal (independent of $\eta$) power-law function of the form

$$G = A \left( \text{Re}_T^{**} \right)^m. \tag{11}$$

Multiplying both parts of (8) by $G$ and carrying out the known transformations ([5]), we obtain

$$\frac{df_T}{dr} = f_T \left( \frac{\eta'}{\eta} + \frac{m}{r} \right) + F_T \eta \left( \eta' = \frac{d\eta}{dr} \right), \tag{12}$$

where $F_\tau = (m+1)(\chi - f_\tau)$,

$$\chi = \frac{\mathrm{Nu}\, G}{F_\tau \,\mathrm{Re}} \tag{13}$$

is the function $f_\tau$. If $\chi$ is normalized so that $\chi = 1$ for $T_{\mathrm{st}} = r^2$, then, by analogy with the dynamic boundary layer, one may assume that the function

$$F_\tau = (m+1)(1 - f_\tau) \tag{14}$$

will retain its form for a broad class of distributions $T_{\mathrm{st}}(r)$. Then

$$f_\tau = \eta T_{\mathrm{st}}^{-(m+1)} r^{-(m+2)} \left[ (m+1) \int_0^r T_{\mathrm{st}}^{m+1} r^{m+2}\, dr + C \right], \tag{15}$$

where $C = 0$ from the condition of finiteness of $f_\tau$ for $T_{\mathrm{st}} = r^2$ and $r = 0$. As a result, for $\mathrm{Re}_\tau^{**}$ we obtain the formula

$$(\mathrm{Re}_\tau^{**})^{m+1} = \frac{m+1}{A}\,\mathrm{Re}\, T_{\mathrm{st}}^{-(m+1)} r^{-(m+3)} \int_0^r T_{\mathrm{st}}^{m+1} r^{m+2}\, dr. \tag{16}$$

In this case, from the condition $\chi = 1$ for $T_{\mathrm{st}} = r^2$, by formulas (4) and (5) we obtain, for $\mathrm{Pr} = 1$, in the laminar regime ($\mathrm{Re} < 3 \cdot 10^5$) $m = 1$, $A = 10.52$, and for the turbulent regime $m = 1/4$, $A = 124.2$. Thus the problem is solved. In particular, for $T_{\mathrm{st}} = r^n$ we obtain, at $\mathrm{Pr} = 1$, for the laminar regime

$$\mathrm{Nu} = 0.308\sqrt{(n+2)\,\mathrm{Re}}; \tag{17}$$

for the turbulent regime

$$\mathrm{Nu} = 0.0212\,(n+2.6)^{0.2}\mathrm{Re}^{0.8}. \tag{18}$$

Let us compare solution (17) for power-law distributions $T_{\mathrm{st}} = r^n$ with the exact solution of equation (2), which in the laminar regime takes the form

$$\theta''(\zeta) = H(\zeta)\theta'(\zeta) + nF(\zeta)\theta(\zeta), \tag{19}$$

where $F(\zeta) = v_r/r\omega$, $G(\zeta) = v_\varphi/r\omega$, $H(\zeta) = v_z/\sqrt{\nu\omega}$, $\theta(\zeta) = T/r^n$, $\zeta = z\sqrt{\omega/\nu}$.

Solving equation (19) by the method of successive approximations gives, for the coefficient $\mathrm{Nu}/\sqrt{\mathrm{Re}} = -\theta'(0)$, the values $-0.02$; $0.399$; $0.52$; $0.616$; $0.67$, respectively, for $n = -2$; 0; 1; 2; 3. Formula (17) gives rather close values

0; 0.436; 0.547; 0.616; 0.688, which testifies to the effectiveness of the approximate method.

It is obvious that, for an arbitrary temperature gradient, the effect of blowing will likewise be expressed by formula (6).

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## REFERENCES

1. A. N. Tifford, Scheng To Chu, Proc. Second US Nat. Congr. Appl. Mech., 1955, p. 793.
2. W. G. Cochran, Proc. Cambr. Phil. Soc., 30, 365 (1934).
3. L. A. Dorfman, Izv. AN SSSR, OTN, No. 7 (1957).
4. E. Truckenbrodt, Zs. ang. Math. u. Mech., 34, No. 4/5, 150 (1954).
5. L. G. Loitsyanskii, *Mechanics of Liquids and Gases*, 1950.

*Note: Figure translations are in progress. See original paper for figures.*