MATHEMATICS

L. A. AIZENBERG

1958

SovietRxiv

Abstract

Full Text

MATHEMATICS

L. A. AIZENBERG

ON TEMLYAKOV INTEGRALS AND THE BOUNDARY PROPERTIES OF ANALYTIC FUNCTIONS OF TWO COMPLEX VARIABLES

(Presented by Academician M. A. Lavrent' ev on 11 II 1958)

Let D be a bicircular domain with center at the point $(0,0) \in D$ of the space of complex variables (w,z), whose boundary is twice continuously differentiable and analytically convex from outside, and suppose that the curve $|w| = \Phi(|z|)$, corresponding in the "absolute quadrant" to the boundary of D, is convex.

As A. A. Temlyakov recently established, the domain D can be specified as the domain bounded by the nonanalytic hypersurfaces $|w| = r_1(\tau)$, $|z| = r_2(\tau)$, $0 \le \tau \le 1$, where $r_1(0) = 0$, $0 < r_1'(\tau) \le \frac{r_1(\tau)}{\tau}$, $r_1(1) < \infty$,

$$r_2(\tau) = \exp\left[-\int_0^\tau \frac{\tau}{1-\tau}\, d\ln r_1(\tau)\right].$$

A. A. Temlyakov showed ([1],[2]):

1. If $f(w,z)$ is analytic in D and continuous in $\overline{D}$, then in the domain D the following integral representation holds:

$$f(w,z) = \frac{1}{4\pi^2 i}\int_0^{2\pi} dt \int_0^1 d\tau \int_{|\zeta|=1} \frac{\zeta f(r_1(\tau)\zeta,\ r_2(\tau)\zeta e^{-it})}{(\zeta-u)^2}\, d\zeta, \qquad (1)$$

where

$$u = \tau\frac{w}{r(\tau)} + (1-\tau)\frac{ze^{it}}{r_2(\tau)}.$$

2. If $f(w,z)$ is analytic in D and the operator $L[f] \equiv f + wf_w' + zf_z'$ is analytic in D and continuous in $\overline{D}$, then

$$f(w,z)=\frac{1}{4\pi^2 i}\int_0^{2\pi}dt\int_0^1 d\tau\int_{|\zeta|=1}\frac{F(r_1(\tau)\zeta,\ r_2(\tau)\zeta e^{-it})}{\zeta-u}\,d\zeta, \tag{2}$$

where $F(w,z)=L[f]$.

We shall call integral (2) a **Temlyakov integral of the first kind**, and integral (1) a **Temlyakov integral of the second kind**.

We shall say that $F(r_1(\tau)\zeta,\ r_2(\tau)\zeta e^{-it})\in\gamma$ if F is continuous in the aggregate of the arguments (τ,t,ζ) $(0\le\tau\le 1,\ 0\le t\le 2\pi,\ |\zeta|=1)$ and satisfies the condition $|F(\tau,t,\zeta)-F(\tau,t,\zeta_0)|<K|\zeta-\zeta_0|^\alpha$, where $0<\alpha\le 1$, with K and α independent of (τ,t).

Theorem 1. *If $F(\tau,t,\zeta)\in\gamma$, then the integral of Temlyakov type of the first kind*

$$f(w,z)=\frac{1}{4\pi^2 i}\int_0^{2\pi}dt\int_0^1 d\tau\int_{|\zeta|=1}\frac{F(\tau,t,\zeta)\,d\zeta}{\zeta-u} \tag{3}$$

is an analytic function in D, continuous in the closed domain $\overline{D}$.

The proof of Theorem 1 reduces to proving that the formula

$$f(w,z)=\frac{1}{2\pi}\int_0^{2\pi}dt\int_0^1\varphi(w,z,t,\tau)\,d\tau, \tag{4}$$

where

$$\varphi(w,z,t,\tau)=\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{F(\tau,t,\zeta)}{\zeta-u}\,d\zeta \qquad (|u|<1),$$

is valid in the closed domain $\overline{D}$, and moreover $\varphi(w,z,t,\tau)$ is analytic in D for all (t,τ), continuous in $\overline{D}$, and continuous with respect to t and τ.

Definition 1. Domains D, where $r_1(\tau)=\dfrac{\tau}{a}$, $r_2(\tau)=\dfrac{1-\tau}{b}$, $a>0$, $b>0$, i.e. domains of the form $a|w|+b|z|\le 1$, will be called **domains of type A**.

Definition 2. If $\lim_{\tau\to 0}\dfrac{\tau}{r_1(\tau)}=0$ and $\lim_{\tau\to 1}\dfrac{1-\tau}{r_2(\tau)}=0$, then the corresponding domains of type D will be called **domains of type B**.

Theorem 2. Let D be a domain of type A and $F\in\gamma$. Then the Temlyakov integral of the first kind, taken over the boundary D, has the following properties:

a) in the domain $E_1 = \{w, z : \ |a|w| - b|z|| > 1\}$ it is an analytic function, continuous in $\overline{E}_1$;

b) in the domain $E_2 = \{w, z : \ |a|w| - b|z|| < 1, \ a|w| + b|z| > 1\}$ it is a nonanalytic but continuous function.

The proof reduces to determining when $|u| < 1$ and when $|u| > 1$, where

$$u = \tau \frac{w}{r_1(\tau)} + (1-\tau)\frac{z}{r_2(\tau)} e^{it}.$$

Theorem 3. Let $F \in \gamma$ and let D be a domain of type A. Then in the whole space (w, z), except for the skeleton of the domain D $\left(|w| = \frac{1}{a}, \ |z| = 0 \text{ or } |w| = 0, \ |z| = \frac{1}{b}\right)$, the integral (3) is a continuous function. At the points of the skeleton the following formulas hold:

$$f_i(w_0, z_0) = \frac{1}{4\pi^2 i} \int_0^{2\pi} dt \int_0^1 d\tau \int_{|\zeta|=1} \frac{F(\tau, t, \zeta)\, d\zeta}{\zeta - u_0} + \frac{1}{2}\left[\frac{1}{2\pi}\int_0^{2\pi} dt \int_0^1 F(\tau, t, u_0)\, d\tau\right],$$

$$f_e(w_0, z_0) = \frac{1}{4\pi^2 i} \int_0^{2\pi} dt \int_0^1 d\tau \int_{|\zeta|=1} \frac{F(\tau, t, \zeta)\, d\zeta}{\zeta - u_0} - \frac{1}{2}\left[\frac{1}{2\pi}\int_0^{2\pi} dt \int_0^1 F(\tau, t, u_0)\, d\tau\right],$$

where

$$u_0 = \tau \frac{w_0}{r_1(\tau)} + (1-\tau)\frac{z_0}{r_2(\tau)} e^{it},$$

$$f_i(w_0, z_0) = \lim_{\substack{(w,z)\to(w_0,z_0) \\ a|w|+b|z|<1}} f(w, z), \qquad f_e(w_0, z_0) = \lim_{\substack{(w,z)\to(w_0,z_0) \\ |a|w|-b|z||>1}} f(w, z).$$

For the proof of Theorem 3 one applies the formulas of Yu. V. Sokhotski for the function

$$\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{F(\tau, t, \zeta)\, d\zeta}{\zeta - u},$$

integrated with respect to t and τ, and uses formula (4).

Remark 1. If $|u| = 1$, then the inner integral in the expression of the Temlyakov integral of the first kind is understood by us as a singular integral.

Remark 2. The behavior of the integral (3), taken over the boundary of a domain of type A, can be illustrated on the "absolute quadrant plane" (see Fig. 1).

Theorem 4. If $F \in \gamma$, then the Temlyakov integral of the first kind, taken over the boundary of a domain D of type B, is a nonanalytic function outside the domain D, continuous in the whole space (w, z).

The proof follows from the fact that the behavior of $|u|$, $(w,z) \in \overline{D}$, is analogous to the behavior of $|u|$ when $|a|w| - b|z|| < 1$ and $a|w| + b|z| > 1$ in the case of a domain of type A.

Remark 3. From the results obtained it is clear that the Temlyakov integral outside the domain D and on its boundary behaves quite differently from the Martinelli-Bochner, Bergman, and Weil integrals, which behave in the same way as the classical Cauchy integral in the case of one variable ([3–5]).

Fig. 1

Definition 3. We shall say that a function regular in the domain D belongs to the class h_δ if

$$\lim_{\rho \to 1} \frac{1}{4\pi^2} \int_0^{2\pi} dt \int_0^1 d\tau \int_0^{2\pi} \left| f\left(r_1(\tau)\rho e^{iQ},\, r_2(\tau)\rho e^{i(Q-t)}\right) \right|^\delta dQ < \infty.$$

Lemma. In order that a function $f(w,z)$, regular in the domain D, belong to the class h_δ, it is necessary and sufficient that the following conditions be fulfilled:

1) for almost all (t,τ) (in the sense of plane Lebesgue measure), the function $f\left(r_1(\tau)u,\, r_2(\tau)ue^{-it}\right)$ belongs to the class H_δ in the unit disk $|u| < 1$;
2) on the boundary of D, $|f(w,z)|^\delta$ is summable.

The proof is based on the well-known lemma of Fatou and the theorem of Riesz ([6,7]).

With the help of this lemma it is easy to prove theorems analogous to the theorems of V. I. Smirnov, F. Riesz, P. Ya. Polubarinova-Kochina, and to some other theorems on boundary properties of analytic functions of one complex variable ([6,7]).

Theorem 5. If a function $f(w,z)$, regular in the domain D, is representable inside D by means of one of the formulas:

$$f(w,z) = \frac{1}{4\pi^2 i} \int_0^{2\pi} dt \int_0^1 d\tau \int_{|\zeta|=1} \frac{F\left(r_1(\tau)\zeta,\, r_2(\tau)\zeta e^{-it}\right) d\zeta}{\zeta - u}$$

(the Temlyakov integral of the first kind), where $F(w,z) = L[f]$;

$$f(w,z) = \frac{1}{4\pi^2}\int_0^{2\pi} dt \int_0^1 d\tau \int_0^{2\pi} \frac{(1-\rho^2)F\left(r_1(\tau)e^{iQ},\, r_2(\tau)e^{i(Q-t)}\right) dQ}{1+\rho^2-2\rho\cos(Q-\varphi)}$$

(the Poisson–Temlyakov integral of the first kind, where $\rho e^{i\varphi} = u$), then it is also representable by means of the other of the indicated formulas. The class of functions $f(w,z)$ representable by these formulas coincides with the class of such $f(w,z)$ for which $F\left(r_1(\tau)u,\, r_2(\tau)ue^{-it}\right) \in H_1$ in the disk $|u| < 1$ for almost all (t,τ).

The proof proceeds from the lemma and from the well-known theorem of G. M. Fichtenholz ([6]).

From Theorem 5 and the lemma it follows:

Corollary 1. If in Theorem 5 the repeated integrals are replaced by triple integrals, then the class of functions $f(w,z)$ will coincide with the class of such functions $f(w,z)$ that $F(w,z) \in h_1$.

Theorem 6. If a function $F(w,z)$, regular in the domain D, is representable inside D by means of one of the formulas:

$$F(w,z) = \frac{1}{4\pi^2 i}\int_0^{2\pi} dt \int_0^1 d\tau \int_{|\zeta|=1} \frac{\zeta F(r_1(\tau)\zeta,\, r_2(\tau)\zeta e^{-it})\, d\zeta}{(\zeta-u)^2}$$

(a Temlyakov integral of the second kind),

$$F(w,z) = \frac{1}{4\pi^2}\int_0^{2\pi} dt \int_0^1 d\tau \int_0^{2\pi} \frac{1-4\rho^2+4\rho^3\cos(Q-\varphi)-\rho^4}{\left[1+\rho^2-2\rho\cos(Q-\varphi)\right]^2} F\, dQ$$

(a Poisson–Temlyakov integral of the second kind), then it is representable also by the other of the indicated formulas. The class of functions representable by these formulas coincides with the class of such $F(w,z)$ that $F(r_1(\tau)u,\, r_2(\tau)ue^{-it}) \in H_1$ in the circle $|u| < 1$ for almost all (t,τ).

From Theorem 6 and the lemma it follows:

Corollary 2. If in Theorem 6 the repeated integrals are replaced by triple integrals, then the class of functions $F(w,z)$ will coincide with the class h_1.

Theorem 7. Let, for the function

$$F(w,z) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_{m,n} w^m z^n,$$

the domain D be the domain of convergence of the double power series. Then, in order that the function $\varphi(w,z) \equiv F^2(w,z)$ be representable by some repeated

Temlyakov integral of the second kind, it is necessary and sufficient that the series

$$\sum_{j=0}^{\infty}\left|\sum_{m=0}^{j} a_{m,j-m}\left(r_1(\tau)\right)^m\left(r_2(\tau)\right)^{j-m} e^{imt}\right|^2$$

converge for almost all (t, τ).

The proof is based on Theorem 6 and on the theorem on the behavior of the coefficients of functions of the class H_2 of one variable ([6]).

We note that all the propositions of the present paper can be extended to the case of n complex variables.

I express my deep gratitude to Prof. A. A. Temlyakov.

Moscow Regional Pedagogical Institute
named after N. K. Krupskaya

Received
8 II 1958

REFERENCES CITED

[1] A. A. Temlyakov, *Izv. Akad. Nauk SSSR, Ser. Mat.*, No. 21, 89 (1957).
[2] A. A. Temlyakov, *Uch. Zap. Moskovsk. Obl. Ped. Inst.*, 21, 7 (1954).
[3] S. N. Look, T. D. Chung, *Acta Math. Sinica*, 7, 1 (1957).
[4] B. A. Fuks, *Theory of Analytic Functions of Several Complex Variables*, Moscow, 1948.
[5] S. Bergmann, *Matem. sborn.*, 1 (43), 6 (1936).
[6] I. I. Privalov, *Boundary Properties of Analytic Functions*, Moscow, 1950.
[7] G. M. Goluzin, *Geometric Theory of Functions of a Complex Variable*, Moscow, 1952.

Note: Figure translations are in progress. See original paper for figures.