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Abstract

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On a Certain "Exact Estimate" of the Multipliers of Second-Order Differential Equations with Periodic Coefficients

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We shall consider a differential equation of the form

$$y'' + p(x)y = 0 \qquad (-\infty < x < \infty;\ p(x+T) = p(x)). \tag{1}$$

Let $\varphi(x)$ and $\psi(x)$ denote two solutions of this equation, determined by the conditions $\varphi(0) = 1,\ \varphi'(0) = 0,\ \psi(0) = 0,\ \psi'(0) = 1.$ As is known, the quantity

$$A = {}^1/_2[\varphi(T) + \psi'(T)] \tag{2}$$

is called the Lyapunov constant, and the roots of the equation

$$\rho^2 - 2A\rho + 1 = 0, \tag{3}$$

which we shall denote by ρ_1 and ρ_2 ($\rho_1\rho_2 = 1$), are called the multipliers. According to Floquet' s theorem, equation (1) has two solutions $y_1(x)$ and $y_2(x)$ such that

$$y_1(x+T) = \rho_1 y_1(x), \qquad y_2(x+T) = \rho_2 y_2(x) \qquad (-\infty < x < \infty), \tag{4}$$

if $A \neq \pm 1$. If $|A| < 1$, then both multipliers are equal to one in modulus and both solutions (4) are bounded on the whole axis. If, however, $|A| > 1$, then one of the multipliers, say ρ_1, is greater than one in absolute value, and the solution $y_1(x)$ increases exponentially as $x \to \infty$. In this case the rate of growth is characterized by the magnitude of the multiplier ρ_1.

A. M. Lyapunov ([1,2]) considered the family of equations

$$y'' + \lambda p(x)y = 0$$

and showed that the real λ-axis contains an infinite number of intervals within which $A = A(\lambda)$ is less than one in absolute value. These intervals are called zones of stability and are numbered from left to right in such a way that the leftmost interval containing positive λ' s is assigned number 1. Between each pair of adjacent stability zones there lies a zone of instability, which is either an isolated point or a closed interval; moreover, to the right of the first stability zone lies the first instability zone, etc. The function $p(x)$, by definition, belongs to the k-th zone of stability (instability) if the number $\lambda = 1$ belongs to the k-th zone of stability (instability).

1. In the works of V. A. Yakubovich ([3,4]) and V. I. Burdina ([5]), some estimates of the absolute values of the multipliers are given. In the present work an estimate is obtained for the absolute value of a multiplier which, in a number of cases, is more convenient.

Denote by Q_α the totality of all real functions $q(x)$ $(q(x+T) \equiv q(x))$, quadratically integrable on the interval $(0, T)$ and satisfying the conditions:

$$1)\quad T\int_0^T q^2(x)\,dx = \alpha; \qquad 2)\quad \int_0^T q(x)\,dx = 0. \tag{5}$$

The totality of all generalized functions $p(x)$ of the form $p(x) = q'(x)$ $(q \in Q_\alpha)$ will be denoted by P_α. We note that if $p(x) \in P_\alpha$, then $\int_0^T p(x)\,dx = 0$.

Theorem 1. *If the function $p(x) \in P_\alpha$ belongs to an instability zone, then the number n of this zone does not exceed the integer part of the number $[2\sqrt{\alpha}/\pi]$. Whatever $n < [2\sqrt{\alpha}/\pi]$ may be $(n = 1, 2, \ldots)$, there exists a function $p \in P_\alpha$ belonging to the n-th instability zone.*

This assertion is "sharp"in the sense that the number $[2\sqrt{\alpha}/\pi]$ cannot be replaced by a smaller number.

Corollary. *If the function $p(x) \in P_\alpha$ belongs to an instability zone, then the number n of this zone does not exceed the number $[2\sqrt{\alpha}/\pi + 1]$.*

Theorem 2. *If the function $p(x) \in P_\alpha$ belongs to the n-th instability zone $(n = 1, 2, \ldots;\ n \le 2\sqrt{\alpha}/\pi)$, then for the multipliers corresponding to the function $p(x)$ the estimate*

$$|\rho_{1,2}| \le \exp\left[\pi\sqrt{\frac{2n}{\pi}\sqrt{\alpha} - n^2}\right] \tag{6}$$

holds.

This estimate is sharp. It is attained for the function $q = q_n(x)$ $(q_n \in Q_\alpha)$

$$q_n = \frac{\sqrt{\alpha}}{T}\operatorname{ctg}\beta, \qquad x = \frac{T}{n\pi}\left(\beta - \frac{\sin 2\beta}{2}\right) \quad (-\infty < \beta < \infty), \tag{7}$$

since the multipliers corresponding to the function $p_n = q_n'$ are equal to

$$\rho_{1,2} = (-1)^n \exp\left[\pm\pi\sqrt{\frac{2n}{\pi}\sqrt{\alpha} - n^2}\right] \qquad \left(n \le \left[\frac{2\sqrt{\alpha}}{\pi}\right]\right). \tag{8}$$

It is easy to see that the function $p_n(x)$ indeed belongs to the n-th instability zone. In fact, the multiplier $\rho_{1,2}(\lambda)$ corresponding to the function $\lambda p_n = \lambda q_n'$ is equal to

$$\rho_{1,2}(\lambda) = (-1)^n \exp\left[\pm\pi\sqrt{\frac{2n\lambda}{\pi}\sqrt{\alpha} - n^2}\right].$$

When λ runs through the values from 0 to 1, the multiplier $\rho_1(\lambda)$ first describes an arc of the unit circle of length $n\pi$, and then becomes and remains real.

For the proof of Theorems 1 and 2 the following variational problem was solved.

To each function $p(x) \in P_\alpha$ there corresponds the Lyapunov number $A = A(p)$. It is required to find $\sup A(p)$ over all functions $p(x) \in P_\alpha$ belonging to the n-th instability zone.

2. The totality of functions entering the n-th stability zone and the n-th instability zone will be called the n-th **zone**. There exists a whole series of criteria (for example, the number of zeros on the interval $(0 \le x \le T)$ of solutions of the equation $y'' + p(x)y = 0$) that make it possible to establish whether a function $p(x)$ belongs to the n-th zone. Therefore, a generalization of the stability criterion obtained by M. G. Krein ([6]) is of known practical interest.

Theorem 3. If $p(x) \subset P_\alpha$ belongs to the n-th zone and $\alpha < n^2\pi^2/4$, then all solutions of equation (1) are bounded.

This criterion is "sharp" in the sense that the number $n^2\pi^2/4$ cannot be replaced by any smaller number.

If $\alpha < \pi^2/4$, then the requirement that $p(x)$ belong to the first zone is automatically satisfied.

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Note: Figure translations are in progress. See original paper for figures.