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# ON THE ZEROS OF THE FUNCTION $\zeta(s)$

1958

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**Abstract**

**Full Text**

**MATHEMATICS**

**N. M. KOROBOV**

## **ON THE ZEROS OF THE FUNCTION $\zeta(s)$**

*(Presented by Academician I. M. Vinogradov, 27 VIII 1957)*

In the present paper new estimates of rational trigonometric sums are given, and applications of these estimates to the theory of the Riemann zeta-function and to the question of the distribution of prime numbers are presented. For the zeta-function, a refinement of the estimate of  $|\zeta(s)|$  and an improvement of the boundary of the real part of the zeros of  $\zeta(s)$  are obtained. In the question of the distribution of prime numbers, the corresponding improvement of the estimate of the modulus of the difference  $\pi(x) - \text{li } x$  is given.

Let  $q, a_1, \dots, a_{n+1}$  be integers,  $(q, a_{n+1}) = 1$ , and  $1 < n < p_1 - 1$ , where  $p_1$  is the least prime divisor of  $q$ . Then the following theorem is valid.

**Theorem 1.** Whatever fixed  $\varepsilon$  may be ( $0 < \varepsilon \leq 0.5$ ), there exist an absolute constant  $C$  and a constant  $\alpha = \alpha(\varepsilon)$  such that, for  $P = q^{1/r}$  on the interval  $n + \varepsilon \leq r \leq n + 1 - \varepsilon$ , the estimate

$$\left| \sum_{x=1}^P e^{2\pi i \frac{a_1 x + \dots + a_{n+1} x^{n+1}}{q}} \right| < CP^{1 - \frac{\alpha}{(n \ln n)^{2.5}}}. \quad (1)$$

The estimate (1) on the interval  $n + \varepsilon \leq r \leq n + 1 - \varepsilon$  represents a strengthening of the result indicated in my paper <sup>(1)</sup>. The proof of the theorem, as in <sup>(1)</sup>, is based on a combination of I. M. Vinogradov's method with a new approach to estimates of trigonometric sums, in which the rationality of the sums under consideration is substantially used.

**Theorem 2.** As  $|t| \rightarrow \infty$ , for every fixed  $\varepsilon > 0$  the estimate

$$\zeta(1 + it) = O\{(\ln |t|)^{5/7+\varepsilon}\}$$

is valid.

The proof of the theorem is based on reducing the question of estimating sums of the form

$$\sum_{x=Q+1}^{Q+P} x^{ti}$$

to estimates of rational trigonometric sums satisfying the conditions of Theorem 1.

From Theorem 2, in the usual way <sup>(2,3)</sup>, we obtain the following assertions:

**Theorem 3.** For every fixed  $\varepsilon > 0$  there exists a positive constant  $A = A(\varepsilon)$  such that, in the region

$$\sigma \geq 1 - \frac{A}{(\ln |t|)^{5/7+\varepsilon}},$$

the function  $\zeta(\sigma + it)$  has no zeros.

**Theorem 4.** For every fixed  $\varepsilon > 0$  there exists a positive-constant  $a = a(\varepsilon)$  such that, as  $x \rightarrow \infty$ , the equality

$$\pi(x) = \text{li } x + O\left(xe^{-a\{\ln x\}^{7/12-\varepsilon}}\right).$$

holds.

**Remark.** Using the results of Theorem 1, one can obtain an analogous refinement of estimates in a number of other questions, in particular in questions concerning the bound for the real part of the zeros of Dirichlet  $L$ -functions and the remainder term in the formula for the number of primes  $p \leq x$  belonging to a given arithmetic progression.

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Received  
26 VIII 1957

## REFERENCES

1. N. M. Korobov, DAN, 118, No. 2 (1958).
2. E. K. Titchmarsh, *The Theory of the Riemann Zeta-Function*, IL, 1953.
3. A. E. Ingham, *The Distribution of Prime Numbers*, M.—L., 1936.

*Note: Figure translations are in progress. See original paper for figures.*

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