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Abstract

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SOME THEOREMS ON THE POSSIBILITY OF UNIFORM APPROXIMATION OF CONTINUOUS FUNCTIONS BY ANALYTIC FUNCTIONS

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In questions concerning the study of sets on which all continuous complex functions can be uniformly approximated with arbitrary accuracy by rational functions, the latest substantial results belong to S. N. Mergelyan (^{1}). In order that on a set every continuous function could be uniformly approximated by rational functions, it is necessary that this set be nowhere dense and, moreover, in every domain the complement of the set must be intensive in some sense.

S. N. Mergelyan proved, for example, that if the intersection of the complement of a closed set with every circle of radius r contains at least one component of diameter not less than $r^{3/2-\varepsilon}$ ($\varepsilon > 0$ and does not depend on r), then every function continuous on this set is approximated by rational functions. In this note, in terms of analytic capacity of sets, some sufficient conditions are formulated (and, in terms of harmonic capacity, some necessary conditions) on a set under which every function continuous on this set is approximated by analytic functions.

Notation: τ is the plane of the complex variable $z = x+iy$; Ce is the complement of the set $e \subset \tau$ in the whole plane τ; $\gamma(e)$ is the analytic capacity of the set e (^{2}); $C(e)$ is the harmonic capacity of the set e; σ_z^r is the open circle of radius r with center at the point z; $P_z^r(e) = \frac{1}{r}\gamma(e \cap \sigma_z^r)$ is the mean analytic density of the set e in the circle σ_z^r; $P_z(e) = \inf_{r\to 0} P_z^r(e)$ is the analytic density of the set e at the point z; $\tilde{e}$ is the boundary of the set e; $d(e)$ is the one-dimensional diameter of the set e; $\tilde{e}_r$ is the set of points of τ whose distance from $\tilde{e}$ is not more than r; $h_1(e)$ is the Hausdorff length of the set e.

§ 1. **The basic lemma.**

Lemma 1. Let e be a subset of the circle σ_z^r with radius $r < 1$. Then for every natural k one can specify a function $g_k(\xi)$, analytic outside the set e, bounded (in modulus) everywhere outside the set e by the constant $B_k'/\gamma(e)$, and such that, for $|\xi - z| \geqslant 3r$, the inequality

$$\left|\frac{1}{\xi - z} - g_k(\xi)\right| \leqslant B'_k \frac{r^k}{|\xi - z|^{k+1}},$$

holds, where B'_k, B''_k are constants depending only on k.

Indeed, from the definition of analytic capacity it follows that there exists a function $\varphi(\xi)$, analytic outside the set e, bounded (in ···

modulo) a unit and such that $\lim_{\xi \to \infty} \xi\varphi(\xi) = \frac{1}{2}\gamma(e)$. In a neighborhood of the infinitely remote point $\varphi(\xi)$ expands in the series

$$\varphi(\xi) = \sum_{k=1}^{\infty} \frac{c_k}{(\xi - z)^k},$$

and

$$|c_k| \leq A\gamma(e)[d(e)]^{k-1}$$

(see Theorem 1 (^{2})). Taking, for example, for $k = 2$

$$g_2(\xi) = \frac{2\varphi(\xi)}{\gamma(e)} - 4c_2 \left(\frac{\varphi(\xi)}{\gamma(e)}\right)^2,$$

it is not difficult to verify that this function satisfies the conditions of Lemma 1.

Lemma 2. Let e, e^* be two sets from τ such that $d(\tilde{e}) < 1$, and for every $z \in \tilde{e}$

$$P_z^r(e^*) \geq \alpha r,$$

and let $f(z)$ be a function defined in the whole plane, having modulus of continuity $\omega(\delta)$ and analytic at every point of the set $e \cap Ce_r$. Then there exists a function $f_r(z)$, analytic in some neighborhood of the set e, which uniformly approximates the function $f(z)$ on the set e with accuracy up to

$$C_1\omega(r)\left(1 + \frac{1}{\alpha r}\right),$$

where C_1 is an absolute constant.

Proof. Fix a system of points $z_i \in \tilde{e}$ $(i = 1, 2, \dots, n)$, which form an ε-net $(\varepsilon = 8r)$ for the set $\tilde{e}$ and are mutually separated by at least $6r$. In the circle $\sigma_{z_i}^r$ fix a closed set $f_i \subset e^*$ such that

$$\gamma(f_i) = rP_{z_i}^r(f_i) \geq \frac{1}{2}\alpha r^2.$$

Put

$$g = e + \sum_{i=1}^{n} \left(\sigma_{z_i}^{4r} - f_i\right).$$

Put

$$g = e + \sum_{i=1}^{n} \sigma_{z_i}^{4r}.$$

From the results of S. N. Mergelyan (¹) it follows that there exists a function

$$f^*(z) = f_1(z) + \frac{1}{2\pi} \int_g \frac{\psi_r(\xi)}{\xi - z} \, d\vartheta \, d\eta \qquad (\xi = \vartheta + i\eta),$$

having the following properties:

1. For $z \in g$, $|f(z) - f^*(z)| \leq C_2 \omega(r)$.
2. The function $f_1(z)$ is analytic inside the domain g.
3. $\psi_r(\xi)$ is continuous and

$$\max_{\xi \in g} \psi_r(\xi) \leq C_3 \frac{\omega(r)}{r}.$$

4. If $f(z)$ is analytic in the circle σ_z^*, then $\psi_r(z) = 0$.

Fix a system of pairwise disjoint sets g_i $(i = 1, 2, \dots, n)$ such that

$$\sum_{i=1}^{n} g_1 \subset e_{4r}, \qquad g_i \subset \sigma_{z_i}^{3r} \quad \text{and} \quad d(g_i) \leq 8r \quad (i = 1, 2, \dots, n).$$

Denote by $\varphi_i(z)$ the function $g_k(z)$ $(k = 2)$ (see Lemma 1) corresponding to the set f_i, and by $\varphi(\xi, z)$ the function which, for $\xi \in g_i$, is identically equal to $\varphi_i(z)$. Put

$$f_r(z) = f_1(z) + \frac{1}{2\pi} \int_g \psi_r(\xi) \varphi(\xi, z) \, d\vartheta \, d\eta.$$

For definiteness, assuming that $z \in g_i$, we have

$$|f(z) - f_r(z)| \leq C_2 \omega(r) + \frac{1}{2\pi} \int_g \left| \psi_r(\xi) \left[\frac{1}{\xi - z} - \varphi(\xi, z) \right] \right| d\vartheta \, d\eta \leq$$

$$\leq C_2 \omega(r) + \frac{C_3 \omega(r)}{2\pi r} \int_{g \cap e_{4r}} \left| \frac{1}{\xi - z} - \varphi(\xi, z) \right| d\vartheta \, d\eta \leq$$

$$\leq C_2 \omega(r) + \frac{C_3 \omega(r)}{2\pi r} \sum_{i=1}^{n} \int_{g_i - f_i} \left| \frac{1}{\xi - z} - \varphi(\xi, z) \right| d\vartheta \, d\eta \leq$$

$$\ll C_2 \omega(r) + \frac{C_3 \omega(r)}{2\pi r} \left[\frac{B_2'(8r)^2}{\gamma(l_1)} + C_4 \int_{|\xi - z| \geqslant 3r} \frac{B_2'' r^2 3r}{|\xi - z|^3} \, d\vartheta \, d\eta \right] \ll C_1 \omega(r) \left(1 + \frac{1}{ar} \right).$$

If, however, $z \in g - \sum_{i=1}^{n} g_i$, then the assertion of the lemma is proved still more simply.

From this lemma, for example, it follows:

Theorem 1. *If the set e_0 contains no condensation points* (^{2}), *then for all z and r*

$$P_z^r(Ce_0) = 1.$$

For the proof of the theorem it is enough to use Lemma 2 and Theorem 2 (^{2}), taking

$$e = \tau - \sigma_z^r \cap Ce_0, \qquad e^* = Ce_0 \cap \sigma_z^r,$$

and as $f(z)$ any function $f_z^{r-\varepsilon}(\xi)$, bounded in the whole plane by the constant

$$\frac{1}{r-\varepsilon}$$

and equal, for $|\xi - z| \geqslant r - \varepsilon$, to the function

$$\frac{1}{\xi - z}.$$

§ 2. Approximation of continuous functions by analytic functions.

From Theorem 1 and Lemma 2 we obtain:

Theorem 2. *If the set e has no condensation points, then every function uniformly continuous on this set can be uniformly approximated, with any prescribed accuracy, by a function analytic in some neighborhood of the set e.*

Theorem 3. *If a function $f(z)$ satisfies a Lipschitz condition on the set e and has a derivative with respect to dz at every condensation point of the set e, then it can be expanded on the set e into a uniformly convergent series of analytic functions.*

Theorem 4. *If the set e is closed and at each of its boundary points the analytic density $P_z(Ce) \geqslant \varepsilon$ ($\varepsilon > 0$ does not depend on $z \in \dot{e}$), then, in order that a function $f(z)$ be expandable into a uniformly convergent series of rational functions, it is necessary and sufficient that this function be continuous on e and have a derivative at every interior point of the set e.*

Let us single out one special case of Theorem 2. If the set e is closed and, for all z and r, the set $Ce \cap \sigma_z^r$ contains at least one component with diameter

$$d \geqslant \frac{Br^2}{1 + |\lg r|}$$

(B does not depend on r), then the set e contains no condensation points and, consequently, every function continuous on this set is expandable into a series of rational functions. On the other hand, for every $\varepsilon > 0$ one can indicate a continuum e such that, for all z and r, the set $Ce \cap \sigma_z^r$ has components with diameter

$$d_\varepsilon(r) \geqslant \frac{B_\varepsilon r^2}{1 + |\lg r|^{1+\varepsilon}}$$

($B_\varepsilon > 0$ does not depend either on z or on r), on which the function $f(z) = x$ is not representable in the form of a series of rational functions.

By analogy with Theorem 2 one proves:

Theorem 5. *If the set e is such that for all r and z the harmonic capacity*

$$C(Ce \cap \sigma_z^r) \geqslant \varepsilon r$$

(ε does not depend either on z or on r), then every real function uniformly continuous on the set e can be uniformly approximated, with any prescribed accuracy, by a function harmonic in some neighborhood of the set e.

§ 3. Some necessary conditions for the possibility of approximating continuous functions by analytic functions.

Lemma 3. *If the boundary of a domain g consists of two continuously differentiable components α and β, and $f(x, y)$ is a function harmonic in the domain g and constant on α and β, $f(\alpha) = 0$, $f(\beta) = h$, then*

$$\int_g |\operatorname{grad} f(x, y)|\, dx\, dy \leqslant \operatorname{mes}_2 g + |h| \int_\beta \frac{d}{dn} f(x, y)\, ds.$$

Lemma 4. If g is a domain bounded by two level lines of the Green function $g_e(\infty, z)$ of a certain set e, then

$$\int_g |\operatorname{grad} g_e(\infty, z)|\, dx\, dy \leq \operatorname{mes}_2 g + 2\pi |g_e(\infty, \alpha) - g_e(\infty, \beta)|$$

and, when $g_e(\infty, \alpha) = 0$ and $g_e(\infty, \beta) = 1$, the inequality holds:

$$\int_g |\operatorname{grad} g_e(\infty, z)|\, dx\, dy \leq C_5 C(e)$$

(where $C_5 > 0$ is an absolute constant).

Taking Lemma 4 into account, from A. S. Kronrod' s formula [3] for computing the plane variation

$$V_2[g, f(x, y)] = \int_{\min f(x,y)}^{\max f(x,y)} h_1\{f(x, y) = t\}\, dt = \int_g |\operatorname{grad} f(x, y)|\, dx\, dy$$

and from the fact that every set of zero harmonic capacity has zero length, we obtain:

Lemma 5. For every bounded closed set e and every $\varepsilon > 0$ one can specify an open set $g \supset e$, whose boundary $\gamma_\varepsilon(e)$ consists of a finite number of analytic curves with total length not exceeding $\varepsilon + C_6 C(e)$, where $C_6 > 0$ is an absolute constant.

Theorem 6. If on the set e every complex uniformly continuous function can be represented by a uniformly convergent series of analytic functions, then for all z and r

$$C(\sigma_z^r \cap Ce) \geq C_7 r,$$

where C_7 is an absolute constant.

The idea of the proof is as follows. Since $f_z^{r-\varepsilon}(\xi)$ (see the explanation to Theorem 1) is close to $\dfrac{1}{\xi - z}$ near $\tilde{\sigma}_z^r$, the integral over a contour close to this circle is close to 1; on the other hand, by the condition of the theorem this integral must be approximately equal to the integral of the same function over the contour $\gamma_\varepsilon(Ce \cap \sigma_z^r)$, whose length is commensurable with $C(\sigma_z^r \cap Ce)$; therefore the case $C(\sigma_z^r \cap Ce) = o(r)$ is impossible.

Theorem 7. If on the set e every real function with uniformly continuous partial derivatives can be approximated (in the C_1 metric) with any prescribed accuracy by a harmonic function, then for all z and r

$$C(\sigma_z^r \cap Ce) \geq C_8 r,$$

where C_8 is an absolute constant.

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References

[1] S. N. Mergelyan, *Uspekhi Mat. Nauk*, 7, No. 2 (1952). [2] A. G. Vitushkin, DAN, 123, No. 5 (1958). [3] A. S. Kronrod, *Uspekhi Mat. Nauk*, 5, No. 1 (1950).

Note: Figure translations are in progress. See original paper for figures.

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