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Abstract

Full Text

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FUNDAMENTAL MATRICES OF SOLUTIONS OF GENERAL PARABOLIC SYSTEMS

(Presented by Academician I. G. Petrovskii, 9 XII 1957)

In the author's preceding papers^(3,4), fundamental matrices of solutions (f.m.s.) were constructed and studied for linear systems of first order in t that are parabolic in the sense of I. G. Petrovskii (the maximal order M of differentiation with respect to t coincided with the parabolic weight $2b$). In the present note f.m.s. are constructed for arbitrary linear systems, parabolic in the sense of I. G. Petrovskii,*

$$\frac{\partial^{n_i} u_i}{\partial t^{n_i}} = \sum_{2bk_0 + |k| \leq 2bn_j} A_{ij}^{(k_0 k)}(x, t) \frac{\partial^{k_0}}{\partial t^{k_0}} D_x^k u_j(x, t) \equiv P_i \left(t, x; \frac{1}{i} D, \frac{\partial}{\partial t} \right) u, \quad (1)$$

$$i = 1, 2, \dots, N,$$

and some of their applications are given.

1. By means of the method set forth in⁽⁴⁾, the following theorem is established:

Theorem 1. Suppose: 1) the coefficients of (1) are continuous with respect to t ; moreover, the continuity with respect to t of those coefficients for which $2bk_0 + |k| = 2bn_j$ is uniform in x from $\Pi_1 \{-\infty < x_s < \infty, s = 1, 2, \dots, n, 0 \leq t \leq T\}$; 2) the coefficients of (1) are bounded and Hölder-continuous with respect to x . Then the system (1) has in $\Pi_1, t > \tau$, an f.m.s. $Z_i^{(l)}(t, \tau, x, \xi)$, satisfying the inequalities

$$\left| \frac{\partial^{k_0}}{\partial t^{k_0}} D_x^k Z_i^{(l)}(t, \tau, x, \xi) \right| \leq C(t - \tau)^{-(n+|k|)/2b+n_i-1-k_0} e^{-c\rho}; \quad (2)$$

$$\left| \Delta_h \frac{\partial^{k_0}}{\partial t^{k_0}} D_x^k Z_i^{(l)}(t, \tau, x, \xi) \right| \leq C^* |h|^\alpha (t - \tau)^{-(n+|k|+\alpha)/2b+n_i-1-k_0} e^{-c^*\rho} \quad (3)$$

($2bk_0 + |k| \leq 2bn_i; |h| < a(t - \tau)^{1/2b}$; a is an arbitrary positive number; C, c are positive numbers depending only on T ; C^*, c^* depend on T and a ; $0 < \alpha \leq 1$). If, in addition to condition 1), the following condition is fulfilled: 3) $n_1 = n_2 = \dots = n_N$, the $A_{ij}^{(k_0 k)}(x, t)$ have $k_0 + |k|$ bounded derivatives, Hölder-continuous

with respect to x , with respect to t, x , then $Z'(t, \tau, x, \xi)$, as a function of τ, ξ , is an f.m.s. of the system adjoint to (1). If, moreover, in addition to conditions 1), 2), the following condition is fulfilled:

$$* D_x^k = \frac{\partial^{|k|}}{\partial x_1^{k_1} \partial x_2^{k_2} \dots \partial x_n^{k_n}}; \quad |k| = \sum_{s=1}^n k_s; \quad z = x+iy; \quad x = (x_1, x_2, \dots, x_n); \quad \zeta = \xi+iv;$$

$$q = \frac{2b}{2b-1}; \quad \rho = \sum_{s=1}^n |x_s - \xi_s|^q (t - \tau)^{-1/(2b-1)}; \quad f(x+h) - f(x) = \Delta_h f;$$

δ_{il} is the Kronecker symbol; repetition of an index means that summation is performed over it.

- 4) the coefficients (1) are defined in the domain $G_1 \{ |Z_1 - x_1^0| < \rho_1, -\infty < x_s < \infty, s = 1, 2, \dots, n, 0 \leq t_1 \leq t \leq t_2 \}$ and are in it analytic functions of z_1 , bounded and continuous with respect to x_1, x_2, \dots, x_n, t , moreover the continuity in t of the coefficients for which $2bk_0 + |k| = 2bn_j$ is uniform with respect to (z_1, x_2, \dots, x_n) from G_1^* , then the f. m. s. $Z(t, \tau, x, \xi)$ can be continued into a complex domain G_2 with respect to z_1, ζ_1 in such a way that there it is analytic in the variables z_1, ζ_1 . In this case the estimates

$$\left| D_x^k \frac{\partial^{k_0}}{\partial t^{k_0}} Z_i^{(l)}(t, \tau, x, \xi) \right| \leq C_0 (t - \tau)^{-(n+|k|)/2b+n_i-1-k_0} \times \\ \times \exp\{-c_0 \rho + b_0 |v_1 - w_1|^q (t - \tau)^{-1/(2b-1)}\}, \quad (4)$$

$$2bk_0 + |k| \leq 2bn_i - 1.$$

2. We shall present two lemmas characterizing integral operators whose kernel is the f. m. s., in the spaces L_p with a special weight, on which the study of the Cauchy problem for parabolic systems is based.

Definition. The function $u(x, t)$ belongs to the space $L_{p,k(t)}$, $1 \leq p < \infty$, if the p -th power of the function

$$|u(x, t)| \times \\ \times \exp \left\{ -k(t) \sum_{s=1}^n |x_s|^q \right\}$$

is summable,

$$\|u(x, t)\|_{L_{p, k(t)}} = \left[\int |u(x, t)|^p \exp \left\{ -pk(t) \sum_{s=1}^r |x_s|^q \right\} dx \right]^{1/p}.$$

The function $u(x, t)$ belongs to the space $L_{\infty, k(t)}$, if

$$u(x, t) \times \exp \left\{ -k(t) \sum_{s=1}^n |x_s|^q \right\}$$

is measurable and essentially bounded,

$$\|u(x, t)\|_{L_{\infty, k(t)}} = \sup_x \text{vrai} \left[|u(x, t)| \exp \left\{ -k(t) \sum_{s=1}^n |x_s|^q \right\} \right].$$

By $L_{p, k(t), s}$ we shall denote the space of vector-functions with s components, each of which belongs to $L_{p, k(t)}$;

$$\|u(x, t)\|_{L_{p, k(t), s}} = \sum_{k=1}^s \|u_k\|_{L_{p, k(t)}}.$$

Lemma 1. If $u(x, t) \in L_{p, k(t), N}$; $1 \leq p \leq \infty$;

$$k(t) = \frac{(c - \varepsilon)k}{[(c - \varepsilon)^{2b-1} - k^{2b-1}(t - t_0)]^{\frac{1}{2b-1}}};$$

c —from inequality (2), $0 < \varepsilon < c$; k —an arbitrary positive number for all $t \in [t_0, t_1]$;

$$t_1 = \left(\frac{c - \varepsilon'}{k} \right)^{2b-1}, \quad \varepsilon < \varepsilon' < c,$$

then

$$U_i(x, t) = \int Z_i^{(l)}(t, \tau, x, \xi) u_i(\xi, \tau) d\xi$$

satisfies the estimate

$$\left\| \frac{\partial^{k_0}}{\partial t^{k_0}} D_x^k U_i(x, t) \right\|_{L_{p, k(t)}} \leq C(\varepsilon) (t - \tau)^{n_i - k_0 - 1 - |k|/2b} \|u(x, \tau)\|_{L_{p, k(\tau), N}}, \quad (5)$$

$$2bk_0 + |k| \leq 2bn_j.$$

* It is also assumed that in the domain G_1 the system (1) is parabolic; $t < t_2 \leq T$.

Denote by $M_{p,[t_0,t_1],s} = M_{p,s}$ the space of vector-functions $u(x, t)$ belonging to the space $L_{p,k(t),s}$ and such that the vector-function

$$u(x, t) \exp \left\{ -k(t) \sum_{s=1}^n |x_s|^a \right\}$$

is Bochner integrable (with respect to t , as an element of the space $L_{p,s}$);

$$\|u(x, t)\|_{M_{p,s}} = \int_{t_0}^{t_1} \|u(x, t)\|_{L_{p,k(t),s}} dt.$$

Consider the integral equation

$$v(x, t) = v_0(x, t) + \int_{t_0}^t d\tau \int R(t, \tau, x, \xi) F(\tau, u(\xi, \tau)) d\xi, \quad (6)$$

where

$$R(t, \tau, x, \xi) = \left\| \begin{array}{c} Z(t, \tau, x, \xi) \\ \cdot \\ \cdot \\ \cdot \\ \frac{\partial^{k_0}}{\partial t^{k_0}} D^k Z(t, \tau, x, \xi) \end{array} \right\|, \quad 2bk_0 + |k| \leq m_i \leq 2bn_i - 1,$$

is a matrix having N columns and $r = Ns$ rows, s being equal to the number of derivatives with respect to t, x of order $k_0 + |k|$, $2bk_0 + |k| \leq \max_i m_i$; $F(t, v)$ is an operator defined in the space $M_{p,r}$ with range belonging to $M_{p,N}$; the integral with respect to τ in (6) is understood in the Bochner sense. Denote

$$m = \max_{i=1,2,\dots,N} \{m_i - (n_i - 1)2b\}.$$

Lemma 2. If

$$\|v_0(x, t)\|_{L_{p,k(t),r}} \leq C_0(t - t_0)^{-m/2b};$$

$F(t, v)$ satisfies the conditions:

a)

$$\int_{t_0}^t (t - \tau)^{-m/2b} \|F(\tau, 0)\|_{L_{p,k(\tau),N}} d\tau \leq E_0(t - t_0)^{-m/2b};$$

b)

$$\|F(t, v_1) - F(t, v_2)\|_{L_{p,k(t),N}} \leq A\|v_1 - v_2\|_{L_{p,k(t),r}},$$

then the integral equation (6) has a unique solution $v^*(x, t) \in M_{p,r}$, satisfying the inequality

$$\|v^*(x, t)\|_{M_{p,r}} \leq B(C_0 + E_0).$$

3. We shall study the Cauchy problem for the parabolic system

$$\frac{\partial^{n_i} u_i}{\partial t^{n_i}} = P_i \left(t, x; \frac{1}{i} D, \frac{\partial}{\partial t} \right) u + F_i \left(t, x, u, \dots, \frac{\partial^{k_0}}{\partial t^{k_0}} D^k u_j, \dots \right), \quad (7)$$

$$i = 1, 2, \dots, N, \quad 2bk_0 + |k| \leq m_j,$$

$$\lim_{t \rightarrow +t_0} \left\| \frac{\partial^{k_0} u_i}{\partial t^{k_0}} - \varphi_i(x) \delta_{k_0 n_i - 1} \right\|_{L_{p,k(t)}} = 0, \quad (8)$$

$$k_0 = 0, 1, \dots, n_i - 1, \quad \varphi_i(x) \in L_{p,k(t_0)}.$$

Definition. Let

$$\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_s \end{pmatrix}, \quad sN = r,$$

be a solution of equation (6) with

$$v_0(x, t) = \int R\varphi d\xi,$$

belonging to the space $M_{p,r}$; we shall call the vector-function $v_1(x, t)$ a **generalized solution of problem** (7), (8).

Lemma 2 means, in these terms, that if conditions 1), 2) of Theorem 1 and conditions a), b) of Lemma 2 are satisfied, then problem (7), (8) has a unique generalized solution.

* In the case where $m_i < \max_j m_j$, instead of the derivatives of Z_i of order $k_0 + |k|$,

$2bk_0 + |k| > m_i$, zeros are written.

Theorem 2. Suppose that the following conditions are satisfied: c) the operator $F(t, v) = V(x, t)$ maps all continuous vector-functions (satisfying a Hölder condition in x in any finite domain $G_{at_0^*} \{t_0 < t_0^* \leq t \leq t_1, |x| \leq a\}$) into continuous vector-functions (satisfying a generalized Hölder condition in x); d) for the vector-functions described in condition c),

$$|F(t, v_1) - F(t, v_2)| \leq L|v_1 - v_2|; \quad |F(t, 0)| \leq E_0(t - t_0)^{-m/2b};$$

$\varphi(x)$ is bounded in every finite ball of the space x . Or suppose that the following conditions are satisfied:

$$p > \frac{n}{2b - m}, \quad \int_{t_0}^t \|F(\tau, 0)\|_{L_p, h(\tau), N} (t - \tau)^{-(mp+n)/2bp} d\tau \leq E_0(t - t_0)^{-(mp+n)/2bp}$$

and condition b) of Lemma 2. Then problem (7), (8) has a classical solution $u_i(x, t)$, belonging, together with all its derivatives of order $k_0 + |k|$, $2bk_0 + |k| \leq 2bn_i - 1$, to the space M_p , and depending continuously (in the norm M_p) on $\varphi(x)$ and $F(t, 0)$.

Remark. Under a certain additional smoothness of the coefficients

$$P_i \left(t, x, \frac{1}{i} D, \frac{\partial}{\partial t} \right)$$

the assertion of the theorem is valid for solutions of the Cauchy problem with initial data

$$\lim_{t \rightarrow t_0} \left\| \frac{\partial^{bk_0} u_i}{\partial t^{k_0}} - \varphi_i^{(k_0)}(x) \right\|_{L_p, h(t)} = 0, \quad k_0 = 0, 1, \dots, n_i - 1, \quad i = 1, 2, \dots, N, \quad (8')$$

if $\varphi_i^{(k_0)}(x)$, together with its derivatives with respect to x up to order $(n_i - k_0 - 1)2b$, belongs to the space $L_{p, h(t_0)}$.

Theorem 3. Suppose that the coefficients of system (1) satisfy condition 1) of Theorem 1 and condition: 3') $A_{ij}^{(k_0, k)}(x, t)$ have $k_0^* + |k|$ bounded derivatives with respect to t, x , continuous in the Hölder sense in x ,

$$k_0^* = k_0 + k', \quad k^* = k + k', \quad 2bk_0' + |k'| = (n_1 - n_i)2b, \quad n_1 \geq n_2 \geq \dots \geq n_N.$$

Then problem (1), (8') has no more than one regular solution $u_i(x, t)$, belonging, together with its derivatives up to order $n_i - 1$ with respect to t , to the space M_p , and for which all derivatives of the functions $u_i(x, t)$ with $n_i < n_1$, entering into system (1), satisfy the generalized Hölder condition.

The uniqueness theorem for problem (7), (8) is valid if $F(t, v)$ satisfies condition b) of Lemma 3 in the class of functions $u_i(x, t)$ satisfying the requirements

formulated above and belonging, together with all their derivatives of order $k_0 + |k|$ with respect to t, x ,

$$2bk_0 + |k| \leq m_i,$$

to the space M_p .

4. From the analytic continuability of the matrix $Z(t, \tau, x, \xi)$ and estimates (4) it follows that the regular solutions of system (1) are analytic in the spatial coordinates, if the coefficients are analytic functions of these coordinates. In the case of systems close to linear ones, it is additionally required that F , as a vector-function of $t, x, u, \dots, \frac{\partial^{k_0}}{\partial t^{k_0}} D^k u_j, \dots, 2bk_0 + |k| \leq m_j$, be analytic in all arguments except t .

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