# SOME PROPERTIES OF CHEBYSHEV SETS

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## Abstract

## Full Text

## MATHEMATICS

**N. V. EFIMOV and S. B. STECHKIN**

# SOME PROPERTIES OF CHEBYSHEV SETS

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**1°.** We consider a Banach space $X$ and a certain set $M \subset X$; as usual, we call the number

$$\rho(x, M) = \inf_{y \in M} \|x - y\|$$

the distance from $x \in X$ to $M$.

In connection with the problem of best approximations, we shall call $M$: 1) a **set of existence**, if for every $x \in X$ there is a nearest element $y \in M$, i.e., there is a $y \in M$ such that $\rho(x, M) = \rho(x, y)$; 2) a **set of uniqueness**, if for every $x \in X$ there is at most one nearest element $y \in M$. A set which is simultaneously a set of existence and of uniqueness will be called **Chebyshev**. In what follows $M$ is assumed to be bounded.

**2°.** The present note is devoted mainly to the following question: what are the relations between the class of Chebyshev sets and the class of convex sets? In particular, under what conditions on the Banach space $X$ can one assert that every Chebyshev set $M \subset X$ is convex? Below, a solution of this problem is given in the case of the $n$-dimensional space $X_n$; at the same time, some simple geometric propositions are reported that are valid in an arbitrary $X$.

**3°.** Obviously, every set of existence, and consequently every Chebyshev set, is closed. Therefore in what follows only closed sets are considered.

The simplest examples show that Banach spaces $X$ (or $X_n$) are possible in which there exist Chebyshev nonconvex sets and convex non-Chebyshev sets. Indeed, let $X_2$ be the Euclidean plane on which a new metric is introduced so that the Euclidean square $K$ is taken as the unit circle. Then in $X_2$ there exist convex sets which are not Chebyshev sets, for example $K$; it is clear that the essence of the matter here lies in the presence of rectilinear segments on the unit circle. On the other hand, in $X_2$ there exist Chebyshev sets which are not convex. Such a set is obtained, for example, if from a Euclidean disk we cut out a sector with an obtuse angle whose Euclidean bisector is parallel to a diagonal of the square $K$; here the essence of the matter lies in the presence of angular points on the unit circle.

**4°.** Let $E$ be the unit sphere of the space $X$. The sphere is called **strictly convex** if from the conditions $x_0 \in X$, $x_1 \in X$, $x_0 \neq x_1$, $\|x_0\| = 1$, $\|x_1\| = 1$ it follows that $\|x_t\| < 1$, where $x_t = (1-t)x_0 + tx_1$ $(0 < t < 1)$, i.e., if $E$ contains no segment.

The sphere $E$ **does not contain conical points** (in the two-dimensional case, angular points) if for every $x_0 \in E$ there exists one and only one linear functional $f_0$ satisfying the conditions $\|f_0\| = 1$, $f_0(x_0) = 1$.

These two properties are, in a certain sense, reciprocal. Namely, let $E^*$ be the unit sphere of the conjugate space $X^*$. Then, if $E^*$ is strictly convex, $E$ has no conical points; if $E^*$ has no conical points, then $E$ is strictly convex.

5°. Generalizing the considerations given in 3°, it is easy to see that in any space $X$ in which the sphere $E$ is not strictly convex, there exist convex non-Chebyshev sets; in any space $X$ in which the sphere $E$ contains conical points, there exist Chebyshev but nonconvex sets. Along with this the following theorem is true:

**Theorem 1.** *If in an $n$-dimensional Banach space $X_n$ the unit sphere has no conical points, then every bounded Chebyshev set $M \subset X_n$ is convex.*

If the sphere $E$ is strictly convex, then it is established trivially that every convex set is Chebyshev; hence, on the basis of what was said earlier, as a consequence of Theorem 1 we obtain Theorem 2.

**Theorem 2.** *In an $n$-dimensional Banach space $X_n$ the class of bounded Chebyshev sets coincides with the class of bounded convex sets if and only if the unit sphere $X_n$ is strictly convex and has no conical points.*

The main points of the proof of Theorem 1 are set forth in 7°. In the following paragraph some propositions are given on enlargements of the given set $M$ (they are needed for 7°).

6°. Let $X$ be an arbitrary Banach space, $M \subset X$. The set of points $x \in X$ for which $\rho(x, M) < a$ $(a > 0)$ will be called the $a$-extension of the set $M$ and will be denoted by the symbol $M_a$.

Then:

1) $M_a$ is an open set.
2) If $M$ is connected, then $M_a$ is also connected.
3) If $M$ is bounded, then $M_a$ is also bounded; the boundary of $M_a$ will be denoted by $G_a$, and the set $M_a + G_a$ by $\overline{M}_a$.
4) $G_a$ is the set of all $x \in X$ for which $\rho(x, M) = a$.
5) If $x$ does not belong to $M_a$, then

$$\rho(x, M_a) = \rho(x, \overline{M}_a) = \rho(x, G_a).$$

6) If $x$ does not belong to $M_a$, then

$$\rho(x, M) = \rho(x, M_a) + a.$$

7) $M_{a_1+a_2} = (M_{a_1})_{a_2}$.

8) If $M$ is bounded, then there exists a number $a_0$ such that $G_a$, for every $a > a_0$, has the property of strong star-shapedness, i.e. every ray issuing from some point $p \in X$ intersects $G_a$ exactly once.

9) If $M$ is a set of existence, then $\overline{M}_a$ is also a set of existence.

7°. Let $M \subset X$ be a bounded Chebyshev set. Consider an arbitrary element $x \in X$ not belonging to $M$. By the definition of a Chebyshev set, there exists, and moreover a unique, element $y \in M$ such that $\rho(x, y) = \rho(x, M)$; we shall call this element $y$ the projection of $x$ onto $M$. It is easy to show that every element $x'$ belonging to the segment $[x, y]$ has as its projection onto $M$ the same element $y$. It is not clear in advance whether the remaining elements of the ray that goes from $y$ to $x$ will have a common projection onto $M$. Clarification of this question is essentially the main goal of the subsequent arguments.

Take some number $a > 0$ and a point $x \in X$ lying outside $\overline{M}_a$. We shall call the projection of the point $x$ onto $G_a$ the intersection of $G_a$ with the segment $[x, y]$, where $y$ is the projection of $x$ onto $M$. According to what has been said in this paragraph, taking account of 4) from 6°, we conclude that the projection of $x$ onto $G_a$ is determined uniquely; moreover, if $b > a$, then different points of $G_b$ cannot be projected to one and the same point of $G_a$. Finally, if $M$ is compact, then the projection of $x \in G_b$ onto $G_a$ depends continuously on $x$, and also $x$ depends continuously on its projection. Thus, under the indicated conditions, the projection of $G_b$ onto $G_a$ is homeomorphic to $G_b$. Suppose that $X$ is an $n$-dimensional Banach space-

space. Then, according to 8) and 6°, if $a$ is sufficiently large, then $G_a$ and all $G_b$ $(b > a)$ are $(n-1)$-dimensional topological spheres; therefore the projection of $G_b$ onto $G_a$ covers all of $G_a$ (for any $b > a$). But in this case, as is easy to show, if $x$ does not belong to $\overline{M}_a$ and is projected to the point $y \in M$, then the whole ray going from $y$ to $x$ is projected to the point $y$. A set $M_a$ with this property shall be called a "sun," and the rays along which the points $x \in \overline{M}_a$ are projected onto $M$, its rays. Extend the rays of the "sun" inward into $G_a$ by an amount $a - \alpha$ $(0 < \alpha < a)$; the endpoints of the extended rays form a set $P_\alpha$, which is the projection of $G_a$ into $G_\alpha$ (see 6) and 7) from 6°). $P_\alpha$ is an $(n-1)$-dimensional topological sphere and, consequently, bounds some domain $Q_\alpha$. The domain $Q_\alpha$ is convex, and $P_\alpha$ is its bounding convex surface. Indeed, take an arbitrary point $x_\alpha$ on $P_\alpha$ and consider the ray of the "sun" passing through $x_\alpha$. On this ray, from the outside, take some point $p$ and draw through $x_\alpha$ the sphere $E_\alpha$ with center $p$. The sphere $E_\alpha$ contains no points of $P_\alpha$ inside it (for the proof one should use the definition of a Chebyshev set and take account of 6) from 6°). Let the point $p$ recede along the ray to infinity. If there are no conical points on $E_\alpha$, then the limit of $E_\alpha$ is a hyperplane. This hyperplane is supporting to $P_\alpha$ and, consequently, to $\overline{Q}_\alpha$. Thus, $\overline{Q}_\alpha$ has a supporting plane at each boundary point, i.e., at each point of $P_\alpha$. Hence $\overline{Q}_\alpha$ is a convex body, and $P_\alpha$ is a convex surface. Further, $M \subset Q_\alpha$; consequently, $M$ belongs to the intersection of the $Q_\alpha$. On the other hand, $Q_\alpha \subset \overline{M}_\alpha$. Therefore $M$ is the intersection of the $Q_\alpha$ and, thereby, a convex set. Theorem 1 is proved.

Moscow State University
named after M. V. Lomonosov

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*Note: Figure translations are in progress. See original paper for figures.*