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Abstract

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ON SOME BOUNDARY-VALUE PROBLEMS FOR QUASILINEAR ELLIPTIC SYSTEMS OF FIRST ORDER IN THE PLANE

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We shall write the quasilinear system

$$\begin{aligned}&a_{11}(x,y,u,v)u_x + a_{12}(x,y,u,v)u_y +\\&\quad + b_{11}(x,y,u,v)v_x + b_{12}(x,y,u,v)v_y + d_1(x,y,u,v) = 0,\\&a_{21}(x,y,u,v)u_x + a_{22}(x,y,u,v)u_y +\\&\quad + b_{21}(x,y,u,v)v_x + b_{22}(x,y,u,v)v_y + d_2(x,y,u,v) = 0\end{aligned}$$

in the complex form more convenient for us

$$\frac{\partial w}{\partial \bar z} + \mu_1(z,w)\frac{\partial w}{\partial z} + \mu_2(z,w)\frac{\partial \bar w}{\partial \bar z} + d(z,w) = 0,$$

$$z = x + iy, \qquad w = u + iv, \tag{1}$$

$$\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right).$$

Here μ_1 and μ_2 are certain rational functions of a_{ij}, b_{ij}; $d(z,w)$ is a rational function of a_{ij}, b_{ij}, d_i.

The coefficients of our system are defined when z belongs to a certain simply connected domain G, whose boundary Γ has continuous curvature, and for all w in the complex plane E.

We shall assume that system (1) is uniformly elliptic with respect to all w; in our notation this condition is written in the form

$$|\mu_1(z,w)| + |\mu_2(z,w)| \le \mu_0 < 1. \tag{2}$$

Concerning the coefficients we make the following assumptions:

1. $\mu_i(z,w)$, as functions of z for fixed w, are measurable, and with respect to w satisfy the Lipschitz condition

$$|\mu_i(z,w_1)-\mu_i(z,w_2)| \le K|w_1-w_2|. \tag{3}$$

2. $d(z,w)$, as a function of z for fixed w, belongs to $L_p(G)$, and with respect to w satisfies the condition

$$d(z,w)=d_0(z,w)+d_1(z,w)w+d_2(z,w)\bar{w}, \tag{4}$$

where $\|d_i(z,w)\|_{L_p} < K_1$ uniformly in w, and $d_i(z,w)$ are continuous in w for fixed z $(i=0,1,2)$, $p>2$.

the boundary-value problem

$$\alpha u+\beta v\,\big|_\Gamma = 0, \tag{5}$$

where $\alpha(t),\beta(t)$ are Hölder-continuous on Γ, and $\alpha^2+\beta^2=1$, for the system (1) can always be reduced to the case when the domain G is the circle $|z|\le 1$, and the boundary condition is written in the form [1]

$$\operatorname{Re}\{z^{-n}w(z)\}\,\big|_\Gamma = 0,$$

where n is the index of the problem; it is equal to the change of $\arg\{\alpha(t)+i\beta(t)\}$ under one circuit of the point t around the contour Γ counterclockwise [2].

We shall consider problem (1)–(5) in the following formulations.

The case of nonnegative index $n \ge 0$.

Problem 1. Find a function $w(z)\in W_p^{(1)}(G)$, $p>2$, which satisfies equation (1) and the boundary conditions

$$\operatorname{Re}\{z^{-n}w(z)\}\,\big|_\Gamma = 0, \qquad \int_\Gamma z^{-k}w(z)\,ds = 0 \quad (k=0,1,\dots,2n). \tag{6}$$

The case of negative index $n<0$.

Problem 2. Find a function $w(z)\in W_p^{(1)}(G)$, $p>2$, which satisfies equation (1), and $2|n|-1$ real constants $\lambda_0,\lambda_{\pm1},\dots,\lambda_{\pm|n|-1}$ such that on the boundary the condition

$$\operatorname{Re}\{z^{-n}w(z)\}\,\big|_\Gamma = \operatorname{Re}\left\{\lambda_0+\sum_{k=1}^{|n|-1}(\lambda_k+i\lambda_{-k})z^k\right\}\Bigg|_\Gamma \tag{7}$$

is fulfilled.

Analogously to how this was done for linear systems in [1,3], our problems reduce to an equivalent nonlinear singular integral equation

$$\rho + \mu_1(z, T_n\rho)S_n\rho + \mu_2(z, T_n\rho)\overline{S_n\rho} + d(z, T_n\rho) = 0, \tag{8}$$

where

$$T_n\rho = -\frac{1}{\pi}\iint_G \left[\frac{\rho(\zeta)}{\zeta - z} + \frac{z^{2n+1}\rho(\bar{\zeta})}{1 - z\zeta}\right] dT_\zeta \qquad \text{for } n \geq 0; \tag{9}$$

$$T_n\rho = -\frac{1}{\pi}\iint_G \left[\frac{\rho(\zeta)}{\zeta - z} + \frac{\bar{\zeta}^{\,2|n|-1}\rho(\bar{\zeta})}{1 - z\bar{\zeta}}\right] dT_\zeta \qquad \text{for } n < 0;$$

$$S_n\rho = \frac{\partial}{\partial z}T_n\rho.$$

Let ω be an arbitrary function from $L_p(G)$; consider the linear equation

$$\rho + \mu_1(z, T_n\omega)S_n\rho + \mu_2(z, T_n\omega)\overline{S_n\rho}$$

$$+ d_1(z, T_n\omega)T_n\rho + d_2(z, T_n\omega)\overline{T_n\rho} + d_0(z, T_n\omega) = 0. \tag{10}$$

The solution of this equation exists and is unique; denote it by $\rho = M\omega$ [1]. $M\omega$ is a nonlinear operator.

Lemma 1. *The operator $M\omega$ maps the whole space $L_p(G)$ into a certain ball of finite radius; the latter depends only on μ_0, K_1.*

Proof. In view of conditions (2), (4), uniform with respect to w, the lemma is a consequence of Theorem 1 of [3].

Lemma 2. *The operator $M\omega$ takes a weakly convergent sequence into a weakly convergent one, i.e. the operator $M\omega$ is weakly continuous.*

Proof. Let ω_m converge weakly to ω_0. Then $\rho_m = M\omega$

there is a solution of equation (10) for $\omega = \omega_m$. In view of the weak convergence of ω_m, $\|\omega_m\|_{L_p} \leq K_0$. But $T_n\omega_m$ converges to $T_n\omega_0$ uniformly in G, by virtue of the complete continuity of the embedding operator $W_p^{(1)}(G)$, $p > 2$, in $C(D)$.

By Lemma 1 and the properties of the operators T_{np} and S_{np}, the quantities $\|\rho_m\|_{L_p}$, $\|S_n\rho_m\|_{L_p}$, $\|T_n\rho_m\|$ are bounded in the aggregate.

The difference $(\rho_m - \rho_0)$ satisfies the equation

$$
\begin{aligned}
&(\rho_m - \rho_0) + \mu_1(z, T_n\omega_0)S_n(\rho_m - \rho_0) + \mu_2(z, T_n\omega_0)\overline{S}_n(\rho_m - \rho_0) + \\
&\quad + d_1(z, T_n\omega_0)T_n(\rho_m - \rho_0) + d_2(z, T_n\omega_0)\overline{T}_n(\rho_m - \rho_0) = \\
&= [\mu_1(z, T_n\omega_0) - \mu_1(z, T_n\omega_m)]S_n\rho_m + [\mu_2(z, T_n\omega_0) - \\
&\quad - \mu_2(z, T_n\omega_m)]\overline{S}_n\rho_m + [d_1(z, T_n\omega_0) - d_1(z, T_n\omega_m)]T_n\rho_m + \\
&\quad + [d_2(z, T_n\omega_0) - d_2(z, T_n\omega_m)]\overline{T}_n\rho_m + [d_0(z, T_n\omega_0) - d_0(z, T_n\omega_m)].
\end{aligned} \tag{11}
$$

In view of (3), (4) and what was said above, the right-hand side of equation (11) converges weakly to zero, while the inverse operator of the equation is linear, bounded, and its norm does not depend on m. Therefore $(\rho_m - \rho_0) \to 0$ weakly in $L_p(G)$.

Remark. It can be proved that $M\omega_m$ converges strongly in $L_p(G)$ to $M\omega_0$.

Theorem 1. *There exists at least one solution of Problem 1 (Problem 2).*

Proof. For the solvability of Problem 1 (Problem 2) it is sufficient that the operator $M\omega$ have a fixed point. By Lemmas 1 and 2, applying to the operator $M\omega$ Schauder's principle ([4]) for weakly continuous mappings of functional spaces (the sphere in $L_p(G)$ is weakly compact), we obtain the existence of a solution of equation (9) and, consequently, of equation (8) and of our problems.

Theorem 2. *The solution of Problem 1 (Problem 2) is unique.*

Proof. Suppose that there exist two solutions ρ_1, ρ_2 of Problem 1 (Problem 2); then equation (8) has two solutions. Therefore the difference $\rho_2 - \rho_1$, by virtue of conditions (3), (4), satisfies the linear homogeneous equation

$$
\begin{aligned}
&(\rho_2 - \rho_1) + \mu_1(z, T_n\rho_2)S_n(\rho_2 - \rho_1) + \mu_2(z, T_n\rho_2)\overline{S}_n(\rho_2 - \rho_1) + \\
&\quad + \left\{ \frac{d(z, T_n\rho_2) - d(z, T_n\rho_1)}{T_n\rho_2 - T_n\rho_1} + \frac{\mu_1(z, T_n\rho_2) - \mu_1(z, T_n\rho_1)}{T_n\rho_2 - T_n\rho_1} S_n\rho_1 + \right. \\
&\quad \left. + \frac{\mu_2(z, T_n\rho_2) - \mu_2(z, T_n\rho_1)}{T_n\rho_2 - T_n\rho_1}\overline{S}_n\rho_1 \right\} T_n(\rho_2 - \rho_1) = 0.
\end{aligned}
$$

By the uniqueness theorem for our equation (1), $\rho_2 - \rho_1 = 0$. For the case when $\mu_2(z, w) = 0$, this problem was considered by Nitsche ([5]). In conclusion I express my gratitude to I. N. Vekua, under whose direction this work was carried out.

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Note: Figure translations are in progress. See original paper for figures.